

FUNCTIONAL ANALYTIC BACKGROUND FOR A THEORY OF INFINITE-DIMENSIONAL REDUCTIVE LIE GROUPS

DANIEL BELTIȚĂ

ABSTRACT. Motivated by the interesting and yet scattered developments in representation theory of Banach-Lie groups, we discuss several functional analytic issues which should underlie the notion of infinite-dimensional reductive Lie group: norm ideals, triangular integrals, operator factorizations, and amenability.

1. WHAT A REDUCTIVE LIE GROUP IS SUPPOSED TO BE

Introduction. We approach the problem of finding an appropriate infinite-dimensional version of reductive Lie group. The discussion is motivated by the need to have a reasonably general setting where the representation theoretic properties of the classical Lie groups associated with the Schatten ideals of Hilbert space operators—in the sense of [dlH72]—can be investigated in a systematic way. Thus the theory of operator ideals (see e.g., [GK69], [GK70], [DFWW04], and [KW07]) provides the natural background for the present exposition.

The ideas and methods of representation theory of finite-dimensional Lie groups cannot possibly be extended to the setting of Banach-Lie groups in a direct manner. Any attempt to do that fails because of some phenomena specific to the infinite-dimensional Lie groups: there exist Lie algebras that do not arise from Lie groups (see e.g., [Ne06] or [Be06]), closed subgroups of Lie groups need not be Lie groups in the relative topology (see e.g., [Up85]), one does not know how to construct smooth structures on homogeneous spaces unless one is able to find a direct complement of the Lie subalgebra in the ambient Lie algebra (see [Up85], [BP07], [Ga06], or [Be06]), there exists no Haar measure on topological groups which are not locally compact ([We40]), and finally every infinite-dimensional Banach space is the model space of some abelian Lie group without any non-trivial continuous representations (see [Ba83]).

Nevertheless, the study of representation theoretic properties of some specific Banach-Lie groups has led to a number of interesting results; see for instance the papers [SV75], [Bo80], [Pi90], [Ne98], [NØ98], [BR07] or [BG07]. It seems reasonable to try to find out a class of Banach-Lie groups whose representations can be studied in a coherent fashion following the pattern of representation theory of finite-dimensional reductive Lie groups. The aim of the present paper is to survey some of the ideas and notions that might eventually lead to the description of a class of Banach-Lie groups appropriate for the purposes of such a representation theory. The exposition is streamlined by a number of phenomena that play a central role in the classical theory of reductive Lie groups: Cartan decompositions, Iwasawa decompositions, Harish-Chandra decompositions, and existence of invariant measures. By way of describing appropriate versions of these phenomena in infinite dimensions, the paper provides a self-contained discussion of a number of functional analytic issues which should underlay the notion of infinite-dimensional reductive Lie group: triangular integrals, operator factorizations, and amenability.

Finite-dimensional reductive Lie groups. In order to make clear the structures we shall be looking for in infinite dimensions, we now recall the classical setting. Remarks 1.2 and 1.3 basically concern the matrix Lie algebras/groups, where reductivity means stability under the operation of taking adjoints of matrices. (These remarks will be our main motivation for the discussion of Φ -reductivity in Section 5.) The following general definition is the one used in [Kn96].

2000 *Mathematics Subject Classification.* Primary 22E65; Secondary 22E46, 47B10, 47L20, 58B25.

Key words and phrases. reductive Lie group; group decomposition; amenable group; operator ideal; triangular integral.

Date: February 1, 2008 .

Definition 1.1. A finite-dimensional *reductive Lie group* is actually a 4-tuple (G, K, θ, B) , where G is a finite-dimensional Lie group with the Lie algebra \mathfrak{g} , K is a compact subgroup of G with the Lie algebra \mathfrak{k} , $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ is an involutive automorphism, and $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is a nondegenerate symmetric bilinear form which is $\text{Ad}(G)$ -invariant and θ -invariant, such that the following conditions are satisfied:

- (i) \mathfrak{g} is a reductive Lie algebra with the complexification denoted $\mathfrak{g}^{\mathbb{C}}$;
- (ii) $\mathfrak{k} = \text{Ker}(\theta - 1)$;
- (iii) if we denote $\mathfrak{p} = \text{Ker}(\theta + 1)$, then $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ and we have $B(\mathfrak{k}, \mathfrak{p}) = \{0\}$, B is positive definite on \mathfrak{p} , and negative definite on \mathfrak{k} ;
- (iv) the mapping $K \times \mathfrak{p} \rightarrow G$, $(k, X) \mapsto k \cdot \exp_G X$, is a diffeomorphism;
- (v) for every $g \in G$ the automorphism $\text{Ad}(g): \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ belongs to the connected component of $1 \in \text{Aut}(\mathfrak{g}^{\mathbb{C}})$.

We say that the Lie group G itself is a reductive Lie group if the other items K , θ , and B are clear from the context. \square

Remark 1.2. Let \mathfrak{g} be a real finite-dimensional Lie algebra. Then \mathfrak{g} is a *reductive Lie algebra* if and only if it is isomorphic to a real Lie subalgebra \mathfrak{g}_1 of the matrix algebra $M_n(\mathbb{C})$ for some integer $n \geq 1$ such that for every $X \in \mathfrak{g}_1$ we have $X^* \in \mathfrak{g}_1$. In addition, if \mathfrak{g} were a complex Lie algebra, then \mathfrak{g}_1 could be chosen to be a complex Lie subalgebra of $M_n(\mathbb{C})$. \square

Remark 1.3. Let G be a connected closed subgroup of $\text{GL}(n, \mathbb{C})$ for some $n \geq 1$ such that for every $g \in G$ we have $g^* \in G$. Then G is a reductive Lie group with $K = G \cap \text{U}(n)$, $\theta(X) = -X^*$ for all $X \in \mathfrak{g} = \mathbf{L}(G) \subseteq M_n(\mathbb{C})$ and $B(X, Y) = \text{Re}(\text{Tr}(XY))$ for all $X, Y \in \mathfrak{g}$ (see Example 2 in Section 2 of Chapter VII of [Kn96]).

It is not difficult to prove that coverings with finitely many sheets of any group G as above are reductive Lie groups —the covering groups of this type are precisely the *connected* reductive groups in the sense of Definition 2.5 of [Vo00]. (See Proposition 4.2 below for the way the Cartan decompositions lift to covering groups.) In particular every connected semisimple Lie group with finite center is a reductive Lie group in the sense of Definition 1.1 above (Example 1 in Section 2 of Chapter VII of [Kn96]). \square

Some conventions and notation. Throughout the present paper we denote by \mathcal{H} an infinite-dimensional complex separable Hilbert space, by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on \mathcal{H} and by \mathfrak{F} the two-sided ideal of $\mathcal{B}(\mathcal{H})$ consisting of all finite-rank operators. Some convenient references for infinite-dimensional Lie groups with a functional analytic flavor are [dlH72], [Up85], [Ne04], [Ne06], and [Be06]. As in the latter reference, we shall always denote by $\mathbf{L}(\cdot)$ the Lie functor from the category of Banach-Lie groups into the category of Banach-Lie algebras. We also adopt the convention that Lie groups are denoted by Roman capitals and their Lie algebras are denoted by the corresponding Gothic lower case letters.

2. TRIANGULAR INTEGRALS AND FACTORIZATIONS

Norm ideals. The norm ideals will play a critical role for the present exposition. In fact our candidates for infinite-dimensional reductive groups will be Banach-Lie groups whose model spaces are norm ideals.

Definition 2.1. By *norm ideal* we mean a two-sided ideal \mathfrak{I} of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\|\cdot\|_{\mathfrak{I}}$ satisfying $\|T\| \leq \|T\|_{\mathfrak{I}} = \|T^*\|_{\mathfrak{I}}$ and $\|ATB\|_{\mathfrak{I}} \leq \|A\| \|T\|_{\mathfrak{I}} \|B\|$ whenever $A, B \in \mathcal{B}(\mathcal{H})$. \square

Definition 2.2. Let \widehat{c} be the vector space of all sequences of real numbers $\{\xi_j\}_{j \geq 1}$ such that $\xi_j = 0$ for all but finitely many indices. A *symmetric norming function* is a function $\Phi: \widehat{c} \rightarrow \mathbb{R}$ satisfying the following conditions:

- (i) the function $\Phi(\cdot)$ is a norm on the linear space \widehat{c} and $\Phi((1, 0, 0, \dots)) = 1$;
- (ii) $\Phi(\{\xi_j\}_{j \geq 1}) = \Phi(\{\xi_{\pi(j)}\}_{j \geq 1})$ whenever $\{\xi_j\}_{j \geq 1} \in \widehat{c}$ and $\pi: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$ is bijective.

Any symmetric norming function Φ gives rise to two norm ideals \mathfrak{S}_{Φ} and $\mathfrak{S}_{\Phi}^{(0)}$ as follows. For every bounded sequence of real numbers $\xi = \{\xi_j\}_{j \geq 1}$ define $\Phi(\xi) := \sup_{n \geq 1} \Phi(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) \in [0, \infty]$. For all $T \in \mathcal{B}(\mathcal{H})$ denote

$$\|T\|_{\Phi} := \Phi(\{s_j(T)\}_{j \geq 1}) \in [0, \infty],$$

where $s_j(T) = \inf\{\|T - F\| \mid F \in \mathcal{B}(\mathcal{H}), \text{rank } F < j\}$ whenever $j \geq 1$. With this notation we can define

$$\mathfrak{S}_\Phi^{(0)} := \overline{\mathfrak{F}}^{\|\cdot\|_\Phi} \subseteq \{T \in \mathcal{B}(\mathcal{H}) \mid \|T\|_\Phi < \infty\} =: \mathfrak{S}_\Phi$$

that is, $\mathfrak{S}_\Phi^{(0)}$ is the $\|\cdot\|_\Phi$ -closure of the ideal of finite-rank operators \mathfrak{F} in \mathfrak{S}_Φ . Then $\|\cdot\|_\Phi$ is a norm making \mathfrak{S}_Φ and $\mathfrak{S}_\Phi^{(0)}$ into norm ideals (see §4 in Chapter III in [GK69]). We say that Φ is a *mononormalizing* symmetric norming function if $\mathfrak{S}_\Phi^{(0)} = \mathfrak{S}_\Phi$. If $1 \leq p < \infty$, then the formula $\Phi_p(\xi) = \|\xi\|_{\ell^p}$ whenever $\xi \in \hat{c}$ defines a mononormalizing symmetric norming function and the corresponding norm ideal $\mathfrak{S}_p := \mathfrak{S}_{\Phi_p}$ is the p -th *Schatten ideal*. In the special case $p = 2$ we call \mathfrak{S}_2 the *Hilbert-Schmidt ideal*. \square

Remark 2.3. Every separable norm ideal is equal to $\mathfrak{S}_\Phi^{(0)}$ for some symmetric norming function Φ (see Theorem 6.2 in Chapter III in [GK69]). \square

Remark 2.4. For every symmetric norming function $\Phi: \hat{c} \rightarrow \mathbb{R}$ there exists a unique symmetric norming function $\Phi^*: \hat{c} \rightarrow \mathbb{R}$ such that

$$\Phi^*(\eta) = \sup\left\{\frac{1}{\Phi(\xi)} \sum_{j=1}^{\infty} \xi_j \eta_j \mid \xi = \{\xi_j\}_{j \geq 1} \in \hat{c} \text{ and } \xi_1 \geq \xi_2 \geq \dots \geq 0\right\}$$

whenever $\eta = \{\eta_j\}_{j \geq 1} \in \hat{c}$ and $\eta_1 \geq \eta_2 \geq \dots \geq 0$. The function Φ^* is said to be *adjoint* to Φ and we always have $(\Phi^*)^* = \Phi$ (see Theorem 11.1 in Chapter III in [GK69]). For instance, if $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$, $\Phi_p(\xi) = \|\xi\|_{\ell^p}$ and $\Phi_q(\xi) = \|\xi\|_{\ell^q}$ whenever $\xi \in \hat{c}$, then $(\Phi_p)^* = \Phi_q$. If Φ is any symmetric norming function then the topological dual of the Banach space $\mathfrak{S}_\Phi^{(0)}$ is isometrically isomorphic to \mathfrak{S}_{Φ^*} by the duality pairing $\mathfrak{S}_{\Phi^*} \times \mathfrak{S}_\Phi^{(0)} \rightarrow \mathbb{C}$, $(T, S) \mapsto \text{Tr}(TS)$ (see Theorems 12.2 and 12.4 in Chapter III in [GK69]). \square

The next definition describes the Boyd indices of a symmetric norming function. Some convenient references for this notion are Section 3 in [Ar78] and subsections 2.17–19 in [DFWW04].

Definition 2.5. Let Φ be a symmetric norming function. For each $m \geq 1$ define the linear operators $D_m: \hat{c} \rightarrow \hat{c}$ and $D_{1/m}: \hat{c} \rightarrow \hat{c}$ by

$$D_m \xi = (\underbrace{\xi_1, \dots, \xi_1}_{m \text{ times}}, \underbrace{\xi_2, \dots, \xi_2}_{m \text{ times}}, \dots) \text{ and } D_{1/m} \xi = \left(\frac{1}{m} \sum_{i=1}^m \xi_i, \frac{1}{m} \sum_{i=m+1}^{2m} \xi_i, \dots\right)$$

for arbitrary $\xi = (\xi_1, \xi_2, \dots) \in \hat{c}$. We shall think of D_m and $D_{1/m}$ as linear operators on the space \hat{c} equipped with the norm $\Phi(\cdot)$, so that it makes sense to speak about the norms of these operators. The *Boyd indices* of the symmetric norming function Φ are defined by

$$p_\Phi = \sup_{m \geq 1} \frac{\log m}{\log \|D_m\|} \quad \text{and} \quad q_\Phi = \inf_{m \geq 1} \frac{\log(1/m)}{\log \|D_{1/m}\|}$$

and we have $1 \leq p_\Phi \leq q_\Phi \leq \infty$. We shall say that these indices are *nontrivial* if $1 < p_\Phi \leq q_\Phi < \infty$. \square

Remark 2.6. The Boyd indices of any symmetric norming function Φ are related to the Boyd indices of its adjoint Φ^* by the equations $1/p_\Phi + 1/q_{\Phi^*} = 1/p_{\Phi^*} + 1/q_\Phi = 1$. If $1 < r < \infty$ and $\Phi(\cdot) = \|\cdot\|_{\ell^r}$ then for all $m \geq 1$ we have $\|D_m\| = m^{1/r}$ and $\|D_{1/m}\| = m^{-1/r}$, hence in this case $p_\Phi = q_\Phi = r$. \square

The Boyd indices of symmetric norming functions are important for our present purposes because of the following interpolation theorem.

Theorem 2.7. Let Φ be a mononormalizing symmetric norming function and assume that $1 \leq p < p_\Phi \leq q_\Phi < q \leq \infty$. Then the following assertions hold:

- (a) We have $\mathfrak{S}_p \subseteq \mathfrak{S}_\Phi \subseteq \mathfrak{S}_q$.
- (b) There exists a constant $M_\Phi > 0$ with the following property: If $T: \mathfrak{S}_q \rightarrow \mathfrak{S}_q$ is a bounded linear operator such that $T(\mathfrak{S}_p) \subseteq \mathfrak{S}_p$, then $T(\mathfrak{S}_\Phi) \subseteq \mathfrak{S}_\Phi$ and $\|T|_{\mathfrak{S}_\Phi}\| \leq M_\Phi \max\{\|T|_{\mathfrak{S}_p}\|, \|T|_{\mathfrak{S}_q}\|\}$.

Proof. See Corollary 3.4(i) in [Ar78]. \square

Triangular integrals. The triangular integral is a suitable infinite-dimensional version of the operation of taking the upper triangular part (the lower diagonal part, or the diagonal, respectively) of a square matrix. The classical reference is [GK70] (see also [Er78]). We are going to describe this idea in a setting which is slightly more general than the classical one.

Definition 2.8. Let \mathfrak{B} be a unital associative involutive Banach algebra and \mathcal{I} a *contractive \mathfrak{B} -bimodule*, that is, \mathcal{I} is a Banach space equipped with a trilinear map

$$\mathfrak{B} \times \mathcal{I} \times \mathfrak{B} \rightarrow \mathcal{I}, \quad (b_1, X, b_2) \mapsto b_1 X b_2,$$

and an involutive isometric antilinear map $\mathcal{I} \rightarrow \mathcal{I}$, $X \mapsto X^*$, such that \mathcal{I} is an algebraic \mathfrak{B} -bimodule and in addition $(b_1 X b_2)^* = b_2^* X^* b_1^*$ and $\|b_1 X b_2\| \leq \|b_1\| \|X\| \|b_2\|$ whenever $b_1, b_2 \in \mathfrak{B}$ and $X \in \mathcal{I}$.

Also let \mathcal{E} be a totally ordered set of self-adjoint idempotent elements in \mathfrak{B} such that $0, 1 \in \mathcal{E}$, and denote by $\text{Part}(\mathcal{E})$ the set of all *partitions* of \mathcal{E} , that is, the finite families $\mathcal{P} = \{p_i\}_{0 \leq i \leq n}$ in \mathcal{E} such that $0 = p_0 < p_1 < \dots < p_n = 1$. For such a partition \mathcal{P} of \mathcal{E} we define the *diagonal truncation* operator

$$\mathcal{D}_{\mathcal{P}}: \mathcal{I} \rightarrow \mathcal{I}, \quad \mathcal{D}_{\mathcal{P}}(X) = \sum_{i=1}^n (p_i - p_{i-1}) X (p_i - p_{i-1}),$$

the *strictly upper triangular truncation* operator

$$\mathcal{U}_{\mathcal{P}}: \mathcal{I} \rightarrow \mathcal{I}, \quad \mathcal{U}_{\mathcal{P}}(X) = \sum_{i=1}^n p_{i-1} X (p_i - p_{i-1}),$$

and the *strictly lower triangular truncation* operator

$$\mathcal{L}_{\mathcal{P}}: \mathcal{I} \rightarrow \mathcal{I}, \quad \mathcal{L}_{\mathcal{P}}(X) = \sum_{i=1}^n (p_i - p_{i-1}) X p_{i-1}.$$

Now think of $\text{Part}(\mathcal{E})$ as a directed set with respect to the partial order defined by $\mathcal{P} \leq \mathcal{Q}$ if and only if \mathcal{P} is a subfamily of \mathcal{Q} . If $X \in \mathcal{I}$ and the net $\{\mathcal{U}_{\mathcal{P}}(X)\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ is convergent in \mathcal{I} , then the corresponding limit is denoted by $\mathcal{U}(X)$ and is called the *strictly upper triangular integral* of X . One similarly defines the *strictly lower triangular integral* $\mathcal{L}(X)$ and the *diagonal integral* $\mathcal{D}(X)$ whenever they exist. \square

Proposition 2.9. Let \mathfrak{B} be a unital associative involutive Banach algebra, \mathcal{I} a contractive \mathfrak{B} -bimodule and \mathcal{E} a totally ordered set of self-adjoint idempotents in \mathfrak{B} such that $0, 1 \in \mathcal{E}$. Assume that the following conditions are satisfied:

- (i) Either the Banach space underlying \mathcal{I} is reflexive or the integrals \mathcal{D} and \mathcal{U} are convergent on dense subsets of \mathcal{I} .
- (ii) Both families of operators $\{\mathcal{D}_{\mathcal{P}}\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ and $\{\mathcal{U}_{\mathcal{P}}\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ are uniformly bounded.

Then the following assertions hold:

- (a) The strictly upper triangular integral, the strictly lower triangular integral, and the diagonal integral exist throughout \mathcal{I} and the corresponding mappings are bounded linear idempotent operators $\mathcal{U}, \mathcal{L}, \mathcal{D}: \mathcal{I} \rightarrow \mathcal{I}$.
- (b) There exists the direct sum decomposition

$$\mathcal{I} = \text{Ran } \mathcal{L} \dot{+} \text{Ran } \mathcal{D} \dot{+} \text{Ran } \mathcal{U},$$

and the corresponding decomposition of an arbitrary element $X \in \mathcal{I}$ is $X = \mathcal{L}(X) + \mathcal{D}(X) + \mathcal{U}(X)$. In addition, $\mathcal{L}(X^*) = \mathcal{U}(X)^*$ and $\mathcal{D}(X^*) = \mathcal{D}(X)^*$ for all $X \in \mathcal{I}$.

Proof. If the integrals \mathcal{D} and \mathcal{U} are convergent on dense subsets of \mathcal{I} , then they are convergent everywhere because of hypothesis (ii). Now let us assume that the Banach space underlying \mathcal{I} is reflexive. It is easy to check that for all $\mathcal{P} \in \text{Part}(\mathcal{E})$ all of the operators $\mathcal{L}_{\mathcal{P}}$, $\mathcal{D}_{\mathcal{P}}$, and $\mathcal{U}_{\mathcal{P}}$ are idempotent, their mutual products are equal to 0, and their sum is the identity mapping on \mathcal{I} . In addition, the nets $\{\mathcal{U}_{\mathcal{P}}(X)\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ and $\{\mathcal{L}_{\mathcal{P}}(X)\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ are increasing, while $\{\mathcal{D}_{\mathcal{P}}(X)\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ is decreasing, in the sense that if $\mathcal{P}, \mathcal{Q} \in \text{Part}(\mathcal{E})$ and $\mathcal{P} \leq \mathcal{Q}$, then $\mathcal{U}_{\mathcal{P}}\mathcal{U}_{\mathcal{Q}} = \mathcal{U}_{\mathcal{Q}}\mathcal{U}_{\mathcal{P}} = \mathcal{U}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{P}}\mathcal{L}_{\mathcal{Q}} = \mathcal{L}_{\mathcal{Q}}\mathcal{L}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}$, and

$\mathcal{D}_{\mathcal{P}}\mathcal{D}_{\mathcal{Q}} = \mathcal{D}_{\mathcal{Q}}\mathcal{D}_{\mathcal{P}} = \mathcal{D}_{\mathcal{Q}}$. Now hypotheses (i) and (ii) ensure that Theorem 2.1 of [Er78] applies, whence assertion (a) follows.

For assertion (b) note that $X = \mathcal{L}_{\mathcal{P}}(X) + \mathcal{D}_{\mathcal{P}}(X) + \mathcal{U}_{\mathcal{P}}(X)$ and then take the limit with respect to $\mathcal{P} \in \text{Part}(\mathcal{E})$, for each $X \in \mathcal{I}$. For all $\mathcal{P}, \mathcal{Q} \in \text{Part}(\mathcal{E})$ we have $\mathcal{L}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}} = 0$, whence by taking the limit we get $\mathcal{L}\mathcal{D} = 0$. Similarly, all of the mutual products of the operators \mathcal{L} , \mathcal{D} , and \mathcal{U} , are equal to 0, whence the asserted direct sum decomposition follows. To complete the proof note that for all $X \in \mathcal{I}$ and $\mathcal{P} \in \text{Part}(\mathcal{E})$ we have $\mathcal{L}_{\mathcal{P}}(X^*) = \mathcal{U}_{\mathcal{P}}(X)^*$ and $\mathcal{D}_{\mathcal{P}}(X^*) = \mathcal{D}_{\mathcal{P}}(X)^*$, and then again take the limit with respect to the partition \mathcal{P} of \mathcal{E} . \square

Example 2.10. (See Theorem 4.1 in [Ar78].) Let Φ be a mononormalizing symmetric norming function whose Boyd indices are nontrivial and denote $\mathcal{I} = \mathfrak{S}_{\Phi} \subseteq \mathcal{B}(\mathcal{H})$. Denote by \mathcal{E} the set of orthogonal projections associated with a maximal, totally ordered set of closed subspaces of \mathcal{H} . Then the corresponding strictly upper triangular, the strictly lower triangular, and the diagonal integral exist throughout \mathcal{I} and they define bounded linear idempotent operators $\mathcal{U}, \mathcal{L}, \mathcal{D}: \mathcal{I} \rightarrow \mathcal{I}$ whose mutual products are equal to 0 and $\mathcal{U} + \mathcal{L} + \mathcal{D} = \mathbf{1}$.

Indeed, condition (i) in Proposition 2.9 is satisfied since the integrals of finite-rank operators are clearly convergent. Also, it follows by Theorem 3.2 in [Er78] that the families of operators $\{\mathcal{D}_{\mathcal{P}}\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ and $\{\mathcal{U}_{\mathcal{P}}\}_{\mathcal{P} \in \text{Part}(\mathcal{E})}$ are uniformly bounded on each Schatten ideal \mathfrak{S}_p if $1 < p < \infty$. Now Theorem 2.7 shows that both these families are uniformly bounded on $\mathcal{I} = \mathfrak{S}_{\Phi}$ as well, since the Boyd indices of Φ are non-trivial. Thus condition (ii) in Proposition 2.9 is satisfied as well, and it then follows that the integrals \mathcal{U} , \mathcal{D} , and \mathcal{L} are convergent throughout \mathcal{I} . \square

Triangular factorizations. Our next purpose is to survey some of the methods that allow one to find operator theoretic versions of the well-known LU factorization from linear algebra, that is, the fact that every invertible matrix factorizes as the product of a unitary matrix and a triangular one.

Definition 2.11. Let \mathcal{E} be the set of orthogonal projections associated with a maximal, totally ordered set of closed linear subspaces of \mathcal{H} . Then the *nest algebra* of \mathcal{E} is $\text{Alg } \mathcal{E} = \{b \in \mathcal{B}(\mathcal{H}) \mid (\forall e \in \mathcal{E}) \quad be = ebe\}$, that is, the set of all operators which leave invariant each subspace in the family $\{e(\mathcal{H})\}_{e \in \mathcal{E}}$. \square

Definition 2.12. Let \mathfrak{S}_{I} and \mathfrak{S}_{II} be norm ideals. We shall say that $(\mathfrak{S}_{\text{I}}, \mathfrak{S}_{\text{II}})$ is a pair of *associated* norm ideals if the following conditions are satisfied:

- (i) \mathfrak{S}_{I} is separable;
- (ii) $\mathfrak{S}_{\text{I}} \subseteq \mathfrak{S}_{\text{II}}$;
- (iii) for every maximal, totally ordered set of closed linear subspaces of \mathcal{H} and every $X \in \mathfrak{S}_{\text{I}}$, the corresponding strictly upper triangular integral is convergent in the contractive $\mathcal{B}(\mathcal{H})$ -bimodule $\mathcal{I} = \mathcal{B}(\mathcal{H})$ and $\mathcal{U}(X) \in \mathfrak{S}_{\text{II}}$.

(See [GK70] and [Er72] for the original version of this definition.) \square

Example 2.13. Let Φ be a mononormalizing symmetric norming function whose Boyd indices are non-trivial, and denote $\mathfrak{S}_{\text{I}} = \mathfrak{S}_{\text{II}} = \mathfrak{S}_{\Phi}$. Then Example 2.10 shows that $(\mathfrak{S}_{\text{I}}, \mathfrak{S}_{\text{II}})$ is a pair of associated norm ideals.

As a special case, it follows by Remark 2.6 that each pair $(\mathfrak{S}_p, \mathfrak{S}_p)$ consisting of the p -th Schatten ideal and itself is a pair of associated norm ideals if $1 < p < \infty$. \square

Theorem 2.14. Let $(\mathfrak{S}_{\text{I}}, \mathfrak{S}_{\text{II}})$ be a pair of associated norm ideals and consider the triangular and diagonal integrals on the contractive $\mathcal{B}(\mathcal{H})$ -bimodule $\mathcal{I} = \mathcal{B}(\mathcal{H})$ with respect to the set \mathcal{E} of orthogonal projections associated with some maximal, totally ordered set of closed linear subspaces in \mathcal{H} . Also assume that $0 \leq a \in \text{GL}(\mathcal{H})$. Then the following conditions are equivalent:

- (i) $\mathcal{U}(a^{-1})$ exists and $\mathcal{U}(a^{-1}) \in \mathfrak{S}_{\text{I}}$.
- (ii) $\mathcal{D}(a^{-1})$ exists and $a^{-1} - \mathcal{D}(a^{-1}) \in \mathfrak{S}_{\text{I}}$.

If one of these conditions holds true, then there exist uniquely determined $r \in \mathfrak{S}_{\text{II}} \cap \text{Alg } \mathcal{E}$ and $d = d^* \in \text{Alg } \mathcal{E}$ such that $a = (\mathbf{1} + r)d(\mathbf{1} + r^*)$ and the spectrum of r is equal to $\{0\}$.

Proof. See Theorem 4.2 and Lemma 2.5(i) in [Er72]. \square

Here are two corollaries concerning the group $\mathrm{GL}_\Phi(\mathcal{H})$ of Example 5.2 below.

Corollary 2.15. *Let Φ be a mononormalizing symmetric norming function whose Boyd indices are nontrivial, and denote by \mathcal{E} the set of orthogonal projections associated with a maximal, totally ordered set of closed linear subspaces of \mathcal{H} . If $0 \leq a \in \mathrm{GL}_\Phi(\mathcal{H})$, then there exist uniquely determined operators $d, r \in \mathcal{B}(\mathcal{H})$ such that $0 \leq d \in \mathrm{GL}_\Phi(\mathcal{H}) \cap \mathrm{Alg} \mathcal{E}$, $r \in \mathfrak{S}_\Phi \cap \mathrm{Alg} \mathcal{E}$, the spectrum of r is equal to $\{0\}$, and $a = (\mathbf{1} + r^*)d(\mathbf{1} + r)$.*

Proof. See Theorem 2.14 and Example 2.13. \square

Corollary 2.16. *Assume the setting of Corollary 2.15. Then for every $g \in \mathrm{GL}_\Phi(\mathcal{H})$ there exist the operators $b, u \in \mathrm{GL}_\Phi(\mathcal{H})$ such that $b \in \mathrm{Alg} \mathcal{E}$, $u^*u = \mathbf{1}$, and $g = ub$.*

Proof. Just apply Corollary 2.15 for $a = g^*g$. See for instance the proof of Corollary A.2 in [Be07] for more details. \square

3. INVARIANT MEANS ON GROUPS

Amenable groups. We shall briefly discuss the invariant means on topological groups. These can be thought of as weak versions of Haar measures although they have two main drawbacks: they may not be faithful, in the sense that the mean of a non-zero function with nonnegative values can be equal to zero; and not every Lie group admits an invariant mean. On the other hand, we shall see that many Banach-Lie groups do have invariant means; see for instance Remark 5.6. Classical references for amenability are [Pa88] and [Ey72]. See [Pe06], [Ga06], and [BP07] for some recent developments.

Definition 3.1. Let G be a topological group. Consider the commutative unital C^* -algebra $\ell^\infty(G) = \{\psi: G \rightarrow \mathbb{C} \mid \|\psi\|_\infty := \sup_G |\psi(\cdot)| < \infty\}$ and its automorphisms $L_x, R_x: \ell^\infty(G) \rightarrow \ell^\infty(G)$, $(L_x\psi)(y) = \psi(xy)$ and $(R_x\psi)(y) = \psi(yx)$ whenever $y \in G$ and $\psi \in \ell^\infty(G)$, defined for arbitrary $x \in G$. The space of *right uniformly continuous* bounded functions on G is

$$\mathcal{RUC}_b(G) = \{\psi \in \ell^\infty(G) \mid \text{the map } G \rightarrow \ell^\infty(G), x \mapsto L_x\psi, \text{ is continuous}\}.$$

Similarly, the space of *left uniformly continuous* bounded functions on G is the set $\mathcal{LUC}_b(G)$ of all functions $\psi \in \ell^\infty(G)$ such that the mapping $G \rightarrow \ell^\infty(G)$, $x \mapsto R_x\psi$, is continuous. And the space of *uniformly continuous* bounded functions on G is $\mathcal{UC}_b(G) := \mathcal{RUC}_b(G) \cap \mathcal{LUC}_b(G)$.

We say that the topological group G is *amenable* if there exists a linear functional $\mu: \mathcal{RUC}_b(G) \rightarrow \mathbb{C}$ such that $\mu(\mathbf{1}) = 1$, $0 \leq \mu(\psi)$ if $0 \leq \psi \in \mathcal{RUC}_b(G)$, and $\mu(L_x\psi) = \mu(\psi)$ for all $\psi \in \mathcal{RUC}_b(G)$ and $x \in G$. In this case we say that the linear functional μ is a *left invariant mean* on G . \square

Remark 3.2. If the topological group G is amenable, then every left invariant mean μ is continuous on $\mathcal{RUC}_b(G)$ and $\|\mu\| = 1$. On the other hand, the space $\mathcal{RUC}_b(G)$ is a unital C^* -subalgebra of $\ell^\infty(G)$ which consists only of continuous functions, and it is invariant under the automorphisms L_x for all $x \in G$. Thus the left invariant means on G are precisely the states of the commutative unital C^* -algebra $\mathcal{RUC}_b(G)$ which are invariant under the automorphism group defined by the mappings L_x for arbitrary $x \in G$. \square

Example 3.3. *If the topological group G is either compact or abelian, then it is amenable.* In fact, let $\mathcal{C}(G)$ denote the space of all continuous functions on G . If G is compact then the probability Haar measure defines a linear functional $\mu: \mathcal{C}(G) \rightarrow \mathbb{C}$. Compactness of G implies that $\mathcal{RUC}_b(G) = \mathcal{C}(G)$, and then the basic properties of the Haar measure show that μ is a left invariant mean on G . On the other hand, if G is abelian, denote by G_d the group G endowed with the discrete topology. Then $\mathcal{RUC}_b(G_d) = \ell^\infty(G)$ and it follows by (0.15) in [Pa88] that there exists a left invariant mean $\mu: \ell^\infty(G_d) \rightarrow \mathbb{C}$ on the discrete group G_d . Now the restriction of μ to $\mathcal{RUC}_b(G)$ defines a left invariant mean on G . \square

Example 3.4. *Assume that G is a topological group such that there exists a directed system of amenable topological subgroups $\{G_\alpha\}_{\alpha \in A}$ whose union is dense in G . Then G is amenable.* To see this, we shall say that a linear functional $\mu: \mathcal{RUC}_b(G) \rightarrow \mathbb{C}$ is a *mean* on G if $\mu(\mathbf{1}) = 1$ and $0 \leq \mu(\psi)$ whenever $0 \leq \psi \in \mathcal{RUC}_b(G)$. In this case μ is continuous and $\|\mu\| = 1$. Now for every $\alpha \in A$ denote

$$\Lambda_\alpha = \{\mu \mid \mu \text{ is a mean on } G \text{ and } \mu \circ L_x|_{\mathcal{RUC}_b(G)} = \mu \text{ if } x \in G_\alpha\},$$

which is a w^* -compact subset of the unit ball in the topological dual space $(\mathcal{RUC}_b(G))^*$. In addition, $\Lambda_\alpha \neq \emptyset$. In fact, any left invariant mean μ_α on G_α gives rise to an element $\tilde{\mu}_\alpha \in \Lambda_\alpha$ defined by $\tilde{\mu}_\alpha(\psi) = \mu_\alpha(\psi|_{G_\alpha})$ for all $\psi \in \mathcal{RUC}_b(G)$. On the other hand, if $\alpha, \beta \in A$ and $G_\alpha \subseteq G_\beta$, then $\Lambda_\alpha \supseteq \Lambda_\beta$. Thus $\{\Lambda_\alpha\}_{\alpha \in A}$ is a family of w^* -compact subsets of the unit ball in $(\mathcal{RUC}_b(G))^*$ with the property that the intersection of each finite subfamily is nonempty. Therefore $\bigcap_{\alpha \in A} \Lambda_\alpha \neq \emptyset$, and any element μ in this nonempty intersection is a left invariant mean on G . In fact, we already know that μ is a mean on G . To check that it is left invariant, let $x \in G$ arbitrary. Since the union of the family $\{G_\alpha\}_{\alpha \in A}$ is dense in G there exists a net $\{x_i\}_{i \in I}$ in that union such that $\lim_{i \in I} x_i = x$. Then for every $\psi \in \mathcal{RUC}_b(G)$ we have $\lim_{i \in I} L_{x_i} \psi = L_x \psi$ in $\ell^\infty(G)$, hence $\mu(L_x \psi) = \lim_{i \in I} \mu(L_{x_i} \psi) = \lim_{i \in I} \mu(\psi) = \mu(\psi)$, where the second equality follows since $\mu \in \bigcap_{\alpha \in A} \Lambda_\alpha$ and $x_i \in \bigcup_{\alpha \in A} G_\alpha$ for all $i \in I$. \square

Example 3.5. Let G be a finite-dimensional Lie group with finitely many connected components. Denote by R the radical of G (i.e., the connected subgroup of G corresponding to the largest solvable ideal of the Lie algebra of G) and by G_d the group G endowed with the discrete topology.

- (a) The group G is amenable if and only if it has any of these properties:
 - (i) the group G/R is compact;
 - (ii) there exists no closed subgroup of G isomorphic to the free group \mathbb{F}_2 with two generators;
- (b) The discrete group G_d is amenable if and only if any of the following conditions is satisfied:
 - (j) the group G/R is finite (i.e., G is a solvable Lie group);
 - (jj) there exists no subgroup of G_d isomorphic to \mathbb{F}_2 .

We refer to Theorems (3.8) and (3.9) in [Pa88] for proofs of these facts. \square

Remark 3.6. Assume that $\phi: \tilde{G} \rightarrow G$ is a surjective homomorphism of topological groups such that $\text{Ker } \phi$ is an abelian group. Then the group \tilde{G} is amenable if and only if G is so. In fact, it follows by Example 3.3 that the topological subgroup $\text{Ker } \phi$ of \tilde{G} is amenable, and then the assertion follows for instance by remark 2°) in §3 of Exposé n° 1 of [Ey72]. \square

3.0.1. Mimicking the group algebras of compact groups.

Definition 3.7. Let G be a topological group and consider the duality pairing $\langle \cdot, \cdot \rangle: (\mathcal{RUC}_b(G))^* \times \mathcal{RUC}_b(G) \rightarrow \mathbb{C}$. There exists a bounded bilinear map

$$(\mathcal{RUC}_b(G))^* \times \mathcal{RUC}_b(G) \rightarrow \mathcal{RUC}_b(G), \quad (\mu, \psi) \mapsto \mu \cdot \psi$$

defined by $(\mu \cdot \psi)(x) = \langle \mu, L_x \psi \rangle$ for all $x \in G$ (by (2.11) in [Pa88]). The *Arens-type product* on $(\mathcal{RUC}_b(G))^*$ is the bounded bilinear mapping

$$(\mathcal{RUC}_b(G))^* \times (\mathcal{RUC}_b(G))^* \rightarrow (\mathcal{RUC}_b(G))^*, \quad (\mu, \nu) \mapsto \mu \cdot \nu$$

defined by $\langle \mu \cdot \nu, \psi \rangle := \langle \mu, \nu \cdot \psi \rangle$ for all $\psi \in \mathcal{RUC}_b(G)$.

Similarly, by using the duality pairing $\langle \cdot, \cdot \rangle: (\mathcal{LUC}_b(G))^* \times \mathcal{LUC}_b(G) \rightarrow \mathbb{C}$, one defines a bounded bilinear map

$$(\mathcal{LUC}_b(G))^* \times \mathcal{LUC}_b(G) \rightarrow \mathcal{LUC}_b(G), \quad (\mu, \psi) \mapsto \mu \cdot \psi$$

by $(\mu \cdot \psi)(x) = \langle \mu, R_x \psi \rangle$ for all $x \in G$ (by the version of (2.11) in [Pa88] for left uniformly continuous functions). The *Arens-type product* on $(\mathcal{LUC}_b(G))^*$ is the bounded bilinear mapping

$$(\mathcal{LUC}_b(G))^* \times (\mathcal{LUC}_b(G))^* \rightarrow (\mathcal{LUC}_b(G))^*, \quad (\mu, \nu) \mapsto \mu \cdot \nu$$

defined by $\langle \mu \cdot \nu, \psi \rangle := \langle \nu, \mu \cdot \psi \rangle$ for all $\psi \in \mathcal{LUC}_b(G)$. \square

Remark 3.8. Let G be a topological group, consider the space $\mathcal{C}_b(G)$ consisting of the continuous elements of $\ell^\infty(G)$, and define $\sigma: \mathcal{C}_b(G) \rightarrow \mathcal{C}_b(G)$, $(\sigma(\psi))(x) = \overline{\psi(x^{-1})}$ whenever $x \in G$ and $\psi \in \mathcal{C}_b(G)$. The mapping σ is an antilinear isometric $*$ -endomorphism of the unital C^* -algebra $\mathcal{C}_b(G)$ which satisfies $\sigma^2 = \text{id}_{\mathcal{C}_b(G)}$ and has the following additional properties:

- (1) For every $x \in G$ we have $L_x \circ \sigma = \sigma \circ R_{x^{-1}}$.
- (2) We have $\sigma(\mathcal{RUC}_b(G)) = \mathcal{LUC}_b(G)$ and $\sigma(\mathcal{LUC}_b(G)) = \mathcal{RUC}_b(G)$.

- (3) If we define $\Sigma: (\mathcal{RUC}_b(G))^* \rightarrow (\mathcal{LUC}_b(G))^*$ as the anti-dual map of $\sigma: \mathcal{LUC}_b(G) \rightarrow \mathcal{RUC}_b(G)$, that is, $\langle \Sigma(\mu), \phi \rangle = \overline{\langle \mu, \sigma(\phi) \rangle}$ for all $\mu \in (\mathcal{RUC}_b(G))^*$ and $\phi \in \mathcal{LUC}_b(G)$, then the diagram

$$\begin{array}{ccc} (\mathcal{RUC}_b(G))^* \times \mathcal{RUC}_b(G) & \longrightarrow & \mathcal{RUC}_b(G) \\ \Sigma \times \sigma \downarrow & & \downarrow \sigma \\ (\mathcal{LUC}_b(G))^* \times \mathcal{LUC}_b(G) & \longrightarrow & \mathcal{LUC}_b(G) \end{array}$$

is commutative, where the horizontal arrows are the maps introduced in Definition 3.7.

- (4) The mapping $\Sigma: (\mathcal{RUC}_b(G))^* \rightarrow (\mathcal{LUC}_b(G))^*$ is antilinear, isometric, bijective, and for all $\mu_1, \mu_2 \in (\mathcal{RUC}_b(G))^*$ we have $\Sigma(\mu_1 \cdot \mu_2) = \Sigma(\mu_2) \cdot \Sigma(\mu_1)$.

In fact, property (1) follows by a straightforward computation, and it implies property (2) at once. For property (3) note that for all $\mu \in (\mathcal{RUC}_b(G))^*$ and $\psi \in \mathcal{RUC}_b(G)$ we have

$$\begin{aligned} (\Sigma(\mu) \cdot \sigma(\psi))(x) &= \langle \Sigma(\mu), R_x(\sigma(\psi)) \rangle = \langle \Sigma(\mu), \sigma(L_{x^{-1}}(\psi)) \rangle \\ &= \overline{\langle \mu, \sigma^2(L_{x^{-1}}(\psi)) \rangle} = \overline{\langle \mu, L_{x^{-1}}(\psi) \rangle} = \sigma(\mu \cdot \psi)(x) \end{aligned}$$

whenever $x \in G$. To check property (4), compute

$$\begin{aligned} \langle \Sigma(\mu_2) \cdot \Sigma(\mu_1), \varphi \rangle &= \langle \Sigma(\mu_1), \Sigma(\mu_2) \cdot \varphi \rangle \stackrel{(3)}{=} \langle \Sigma(\mu_1), \sigma(\mu_2 \cdot \sigma(\varphi)) \rangle \\ &= \overline{\langle \mu_1, \mu_2 \cdot \sigma(\varphi) \rangle} = \overline{\langle \mu_1 \cdot \mu_2, \sigma(\varphi) \rangle} = \langle \Sigma(\mu_1 \cdot \mu_2), \varphi \rangle \end{aligned}$$

for every $\varphi \in \mathcal{LUC}_b(G)$. □

The facts described in the following theorem can be found in Theorems 2.5 and 4.1 of [Gr].

Theorem 3.9. *Every topological group G has the following properties:*

- (a) *The Arens-type products make $(\mathcal{RUC}_b(G))^*$ and $(\mathcal{LUC}_b(G))^*$ into associative Banach algebras.*
- (b) *Each continuous unitary representation $\pi: G \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ gives rise to two representations of Banach algebras, $\widehat{\pi}_{\mathcal{R}}: (\mathcal{RUC}_b(G))^* \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ and $\widehat{\pi}_{\mathcal{L}}: (\mathcal{LUC}_b(G))^* \rightarrow \mathcal{B}(\mathcal{H}_\pi)$, by means of the formulas*

$$(3.1) \quad (\widehat{\pi}_{\mathcal{R}}(\mu)\xi \mid \eta) = \langle \mu, (\pi(\cdot)\xi \mid \eta) \rangle \text{ and } (\widehat{\pi}_{\mathcal{L}}(\nu)\xi \mid \eta) = \langle \nu, (\pi(\cdot)\xi \mid \eta) \rangle$$

for all $\xi, \eta \in \mathcal{H}$, $\mu \in (\mathcal{RUC}_b(G))^*$, and $\nu \in (\mathcal{LUC}_b(G))^*$. These representations are related by the commutative diagram

$$(3.2) \quad \begin{array}{ccc} (\mathcal{RUC}_b(G))^* & \xrightarrow{\widehat{\pi}_{\mathcal{R}}} & \mathcal{B}(\mathcal{H}_\pi) \\ \Sigma \downarrow & & \downarrow S \\ (\mathcal{LUC}_b(G))^* & \xrightarrow{\widehat{\pi}_{\mathcal{L}}} & \mathcal{B}(\mathcal{H}_\pi) \end{array}$$

where $S: \mathcal{B}(\mathcal{H}_\pi) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$, $b \mapsto b^*$.

Proof. Assertion (a) follows at once by (2.8) and (the left-sided version of) (2.11) in [Pa88]. See also Example (19.23)(b) in [HR63], [Te63], and [Bu50].

For assertion (b), firstly note that the matrix coefficients $\psi_{\xi, \eta} = (\pi(\cdot)\xi \mid \eta)$ belong to the function space $\mathcal{UC}_b(G) = \mathcal{RUC}_b(G) \cap \mathcal{LUC}_b(G)$ for arbitrary $\xi, \eta \in \mathcal{H}_\pi$. To see this, just note that for all $x \in G$ we have $L_x(\psi_{\xi, \eta}) = \psi_{\xi, \pi(x)^*\eta}$ and $R_x(\psi_{\xi, \eta}) = \psi_{\pi(x)\xi, \eta}$, and then use the continuity of the representation π . Thus the right-hand sides of both equalities in (3.1) make sense, and then by means of the estimate $\|\psi_{\xi, \eta}\|_\infty \leq \|\xi\| \|\eta\|$ we get $\widehat{\pi}_{\mathcal{R}}(\mu), \widehat{\pi}_{\mathcal{L}}(\nu) \in \mathcal{B}(\mathcal{H}_\pi)$ and $\|\widehat{\pi}_{\mathcal{R}}(\mu)\| \leq \|\mu\|$ and $\|\widehat{\pi}_{\mathcal{L}}(\nu)\| \leq \|\nu\|$. In addition, since $\sigma(\psi_{\xi, \eta}) = \psi_{\eta, \xi}$, we get $(\widehat{\pi}_{\mathcal{L}}(\Sigma(\mu))\xi \mid \eta) = \langle \Sigma(\mu), \psi_{\xi, \eta} \rangle = \overline{\langle \mu, \sigma(\psi_{\xi, \eta}) \rangle} = \overline{\langle \mu, \psi_{\eta, \xi} \rangle} = (\widehat{\pi}_{\mathcal{R}}(\mu)\eta \mid \xi) = (\widehat{\pi}_{\mathcal{R}}(\mu)^*\xi \mid \eta)$, whence $\widehat{\pi}_{\mathcal{L}}(\Sigma(\mu)) = \widehat{\pi}_{\mathcal{R}}(\mu)^*$, and thus the diagram (3.2) is commutative. To conclude the proof of (b), let $\mu_1, \mu_2 \in (\mathcal{RUC}_b(G))^*$. Then $(\widehat{\pi}_{\mathcal{R}}(\mu_1 \cdot \mu_2)\xi \mid \eta) = \langle \mu_1 \cdot \mu_2, \psi_{\xi, \eta} \rangle = \langle \mu_1, \mu_2 \cdot \psi_{\xi, \eta} \rangle = \langle \mu_1, \psi_{\widehat{\pi}_{\mathcal{R}}(\mu_2)\xi, \eta} \rangle = (\widehat{\pi}_{\mathcal{R}}(\mu_1)\widehat{\pi}_{\mathcal{R}}(\mu_2)\xi \mid \eta)$, where the next-to-last equality holds since for every $x \in G$ we have $(\mu_2 \cdot \psi_{\xi, \eta})(x) = \langle \mu_2, L_x(\psi_{\xi, \eta}) \rangle = \langle \mu_2, \psi_{\xi, \pi(x)^*\eta} \rangle = (\widehat{\pi}_{\mathcal{R}}(\mu_2)\xi \mid \pi(x)^*\eta) = (\pi(x)\widehat{\pi}_{\mathcal{R}}(\mu_2)\xi \mid \eta) = \psi_{\widehat{\pi}_{\mathcal{R}}(\mu_2)\xi, \eta}(x)$. Thus $\widehat{\pi}_{\mathcal{R}}$ is an algebra representation. Similarly, for $\nu_1, \nu_2 \in (\mathcal{LUC}_b(G))^*$ we can check

that $\nu_1 \cdot \psi_{\xi, \eta} = \psi_{\xi, \widehat{\pi}_{\mathcal{L}}(\nu_1)^* \eta}$, whence as above we get $(\widehat{\pi}_{\mathcal{L}}(\nu_1 \cdot \nu_2)\xi \mid \eta) = (\widehat{\pi}_{\mathcal{L}}(\nu_1)\widehat{\pi}_{\mathcal{L}}(\nu_2)\xi \mid \eta)$, and the proof ends. \square

Remark 3.10. In the setting of Theorem 3.9, $\mathcal{RUC}_b(G)$ and $\mathcal{LUC}_b(G)$ are commutative unital isomorphic C^* -algebras, hence there exist $*$ -isomorphisms $\mathcal{RUC}_b(G) \simeq \mathcal{C}(\mathfrak{M}_0(G)) \simeq \mathcal{LUC}_b(\mathfrak{M}_0(G))$, where $\mathfrak{M}_0(G)$ is a compact topological space. In the special case when the group G is compact we have $\mathfrak{M}_0(G) = G$, $\mathcal{C}(G) = \mathcal{RUC}_b(G) = \mathcal{LUC}_b(G) = \mathcal{UC}_b(G)$, and the involutive Banach algebra $(\mathcal{UC}_b(G))^*$ is the convolution measure algebra of G . In the general case, the topological duals of $\mathcal{RUC}_b(G)$ and $\mathcal{LUC}_b(G)$ still consist of the (not necessarily positive) measures on the compact space $\mathfrak{M}_0(G)$, however the Arens-type products may not be defined by convolution formulas by reasonable convolution formulas involving $\mathfrak{M}_0(G)$ (cf. the comment preceding Proposition (2.25) in [Pa88]). See however the general theory of convolutions of functionals developed in Chapter 5 of [HR63] or the other references mentioned in connection with the proof of Theorem 3.9(a) above. And another problem to be dealt with is to find a method to distinguish in the set of pairs of representations of the Banach algebras $(\mathcal{RUC}_b(G))^*$ and $(\mathcal{LUC}_b(G))^*$ that make the diagram (3.2) commutative, the ones that come from unitary representations of G by means of (3.1). In this connection, let us recall that there exists a promising approach to an axiomatic theory of group algebras by means of the host algebras, that is, C^* -algebras whose representations correspond in a one-to-one fashion with the unitary representations of a given group; see [Gr05], [GN07], and the references therein. \square

Remark 3.11. Let G be an amenable topological group and pick a left invariant mean $\mu: \mathcal{RUC}_b(G) \rightarrow \mathbb{C}$. Then μ is a state of the C^* -algebra $\mathcal{RUC}_b(G)$, and the corresponding Gelfand-Naimark-Segal construction leads to a cyclic $*$ -representation $\iota_\mu: \mathcal{RUC}_b(G) \rightarrow \mathcal{B}(\mathcal{H}^{(\mu)})$. Recall that the Hilbert space $\mathcal{H}^{(\mu)}$ is obtained out of $\mathcal{RUC}_b(G)$ as a quotient followed by a completion with respect to the non-negative definite, sesquilinear form $\mathcal{RUC}_b(G) \times \mathcal{RUC}_b(G) \rightarrow \mathbb{C}$, $(\psi, \chi) \mapsto \mu(\psi\chi^*)$. For each $x \in G$ we have $\mu \circ L_{x^{-1}}|_{\mathcal{RUC}_b(G)} = \mu$, hence the mapping $\mathcal{RUC}_b(G) \rightarrow \mathcal{RUC}_b(G)$, $\psi \mapsto L_{x^{-1}}\psi$ induces a unitary representation $\lambda_\mu: G \rightarrow \mathcal{B}(\mathcal{H}^{(\mu)})$, which is easily seen to be continuous. In the special case when G is compact and μ is the probability Haar measure on G , we have $\mathcal{H}^{(\mu)} = L^2(G, \mu)$ and λ_μ is the regular representation of G . For this reason, in the general case of an amenable group G , one can think of λ_μ as a *regular representation associated with the left invariant mean μ* .

We have to point out that it may happen that $\dim \mathcal{H}^{(\mu)} = 1$. For instance, this is the case if G is an extremely amenable group and the left invariant mean μ is chosen to be multiplicative. (See [Pe06] for specific examples and for details on the latter notions.) It may also happen that the regular representation is trivial, in the sense that $\lambda_\mu(x) = \mathbf{1}$ for all $x \in G$. This is the case for the exotic Banach-Lie groups (see [Ba83]), which are abelian topological groups that do not have any non-trivial continuous representation. It is not difficult to verify that the regular representation λ_μ is trivial if and only if $\mu(\chi\psi) = \mu(\chi L_x\psi)$ for all $\chi, \psi \in \mathcal{RUC}_b(G)$ and $x \in G$. \square

4. LIFTING GROUP DECOMPOSITIONS TO COVERING GROUPS

This section has a technical character and its main purpose is to provide tools for enriching the class of reductive Banach-Lie groups to be set forth in the next section. In the proof of the following statements we use some ideas from the proofs of Theorem 6.31 and 6.46 in [Kn96].

Lemma 4.1. *Assume that G and \tilde{G} are Banach-Lie groups, and $e: \tilde{G} \rightarrow G$ is a covering homomorphism. Let K be any Banach-Lie subgroup of G , denote $\tilde{K} := e^{-1}(K)$, and define $\psi: \tilde{G}/\tilde{K} \rightarrow G/K$, $\tilde{g}\tilde{K} \mapsto e(\tilde{g})K$. Then the following assertions hold:*

- (i) *\tilde{K} is a Banach-Lie subgroup of \tilde{G} and the mapping $e|_{\tilde{K}}: \tilde{K} \rightarrow K$ is a covering homomorphism.*
- (ii) *The mapping ψ is a well-defined diffeomorphism.*
- (iii) *If the group \tilde{G} is connected and the smooth homogeneous space G/K is simply connected, then both \tilde{K} and K are connected.*

Proof. Since $\mathbf{L}(e) = T_1 e: \mathbf{L}(\tilde{G}) \rightarrow \mathbf{L}(G)$ is an isomorphism of Banach-Lie algebras, it follows by Proposition 4.8 in [Be06] that \tilde{K} is a Banach-Lie subgroups of \tilde{G} and the tangent map $\mathbf{L}(e)|_{T_1 \tilde{K}}: T_1 \tilde{K} \rightarrow T_1 K$.

Then Remark C.13(b) in [Be06] shows that $e|_{\tilde{K}}: \tilde{K} \rightarrow X$ is a covering map. This completes the proof of assertion (i).

To prove assertion (ii), note that ψ is injective since $\tilde{K} = e^{-1}(K)$, and ψ is surjective because e is so. On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{e} & G \\ \downarrow & & \downarrow \\ \tilde{G}/\tilde{K} & \xrightarrow{\psi} & G/K \end{array}$$

where the vertical arrows (which are the quotient maps) are submersions. Since the covering map e is a local diffeomorphism, it follows from this commutative diagram that ψ is a local diffeomorphism as well. Then ψ is actually a diffeomorphism since we have already seen that it is bijective.

For assertion (iii), recall from (i) that \tilde{K} is a covering group of K , so it will be enough to show that \tilde{K} is connected. And the latter property follows from the long exact sequence of homotopy groups

$$0 \leftarrow \pi_0(\tilde{G}/\tilde{K}) \leftarrow \pi_0(\tilde{G}) \leftarrow \pi_0(\tilde{K}) \leftarrow \pi_1(\tilde{G}/\tilde{K}) \leftarrow \cdots$$

since $\pi_0(\tilde{G}) = \{0\}$ by the assumption on \tilde{G} , while $\pi_1(\tilde{G}/\tilde{K}) = \{0\}$ by the assumption on G/K along with the fact that G/K is homeomorphic to \tilde{G}/\tilde{K} according to assertion (ii). \square

We now come to a proposition to the effect that the Cartan decompositions lift to the covering groups.

Proposition 4.2. *Let G and \tilde{G} be to Banach-Lie groups, and assume that $e: \tilde{G} \rightarrow G$ is a covering homomorphism. Now let K be any Banach-Lie subgroup of G and denote $\tilde{K} := e^{-1}(K)$. Denote $\mathbf{L}(G) = \mathfrak{g}$ and $\mathbf{L}(K) = \mathfrak{k}$, and assume that \mathfrak{p} is a closed linear subspace of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ and the mapping $\varphi: K \times \mathfrak{p} \rightarrow G$, $(k, X) \mapsto k \cdot \exp_G X$ is a diffeomorphism. Moreover denote $\mathbf{L}(\tilde{G}) := \tilde{\mathfrak{g}}$, $\mathbf{L}(\tilde{K}) := \tilde{\mathfrak{k}}$, and $\tilde{\mathfrak{p}} := \mathbf{L}(e)^{-1}(\mathfrak{p})$. Then the mapping $\tilde{\varphi}: \tilde{K} \times \tilde{\mathfrak{p}} \rightarrow \tilde{G}$, $(\tilde{k}, \tilde{X}) \mapsto \tilde{k} \cdot \exp_{\tilde{G}} \tilde{X}$ is a diffeomorphism as well.*

Proof. First note that \tilde{K} is a Banach-Lie subgroup of \tilde{G} by Lemma 4.1, and $\mathbf{L}(e): \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is an isomorphism of Banach-Lie algebras, so that $\mathbf{L}(e)\tilde{\mathfrak{k}} = \mathfrak{k}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \dot{+} \tilde{\mathfrak{p}}$. Now note that there exists a commutative diagram

$$\begin{array}{ccc} \tilde{K} \times \tilde{\mathfrak{p}} & \xrightarrow{\tilde{\varphi}} & \tilde{G} \\ e|_{\tilde{K}} \times \mathbf{L}(e)|_{\tilde{\mathfrak{p}}} \downarrow & & \downarrow e \\ K \times \mathfrak{p} & \xrightarrow{\varphi} & G \end{array}$$

whose vertical arrows are covering maps (see Lemma 4.1(i)). Since φ is a diffeomorphism by assumption, it then follows that $\tilde{\varphi}$ is a local diffeomorphism. To get the wished-for conclusion, we still have to prove that $\tilde{\varphi}$ is a bijective map.

To check that $\tilde{\varphi}$ is injective, let $\tilde{k}_j \in \tilde{K}_j$ and $\tilde{X}_j \in \tilde{\mathfrak{p}}_j$ for $j = 1, 2$ such that $\tilde{k}_1 \cdot \exp_{\tilde{G}} \tilde{X}_1 = \tilde{k}_2 \cdot \exp_{\tilde{G}} \tilde{X}_2$. By applying the map e to both sides of the latter equality, and using the commutation relation between the exponential maps and group homomorphisms (see e.g., Remark 2.34 in [Be06]), we get $e(\tilde{k}_1) \cdot \exp_G(\mathbf{L}(e)\tilde{X}_1) = e(\tilde{k}_2) \cdot \exp_G(\mathbf{L}(e)\tilde{X}_2)$. Since φ is injective, it then follows that $\mathbf{L}(e)\tilde{X}_1 = \mathbf{L}(e)\tilde{X}_2$ and $e(\tilde{k}_1) = e(\tilde{k}_2)$. The first of these equalities implies that $\tilde{X}_1 = \tilde{X}_2$, whence $\tilde{k}_1 = \tilde{k}_2$ by the assumption on $\tilde{k}_j \in \tilde{K}_j$ and $\tilde{X}_j \in \tilde{\mathfrak{p}}_j$ for $j = 1, 2$. Now, to prove that $\tilde{\varphi}$ is surjective, let $\tilde{g} \in \tilde{G}$ arbitrary. Then $e(\tilde{g}) \in G$ hence there exist $k \in K$ and $X \in \mathfrak{p}$ such that $e(\tilde{g}) = k \cdot \exp_G X$ since φ is surjective. Further on, pick $\tilde{k}_0 \in e^{-1}(k)$ arbitrary and denote $\tilde{X} := \mathbf{L}(e)^{-1}X$. Then $e(\tilde{g}) = e(\tilde{k}_0) \cdot \exp_G(\mathbf{L}(e)\tilde{X}) = e(\tilde{k}_0 \cdot \exp_{\tilde{G}} \tilde{X})$, so that by denoting $\tilde{z} := \tilde{k}_0 \cdot \exp_{\tilde{G}} \tilde{X} \cdot \tilde{g}^{-1}$ we have $\tilde{z} \in e^{-1}(1) \subseteq \tilde{K}$. Thus $\tilde{k} := \tilde{z}^{-1}\tilde{k}_0 \in \tilde{K}$ and we have $\tilde{g} = \tilde{k} \cdot \exp_{\tilde{G}} \tilde{X}$, which concludes the proof. \square

The next proposition shows that the familiar integration of local Cartan involutions to global Cartan involutions carries over to the setting of Banach-Lie groups.

Proposition 4.3. *Let G be a connected Banach-Lie group with the Lie algebra $\mathbf{L}(G) = \mathfrak{g}$, K a Banach-Lie subgroup of G , and assume that there exists an automorphism $\theta \in \text{Aut}(\mathfrak{g})$ such that $\theta^2 = \text{id}_{\mathfrak{g}}$, $\mathbf{L}(K) = \text{Ker}(\theta - \text{id}_{\mathfrak{g}})$, and the mapping $\varphi: K \times \mathfrak{p} \rightarrow G$, $(k, X) \mapsto k \cdot \exp_G X$ is a diffeomorphism. Then there exists a unique automorphism $\Theta \in \text{Aut}(G)$ such that $\mathbf{L}(\Theta) = \theta$ and $K = \{g \in G \mid \Theta(g) = g\}$.*

Proof. Let $e: \tilde{G} \rightarrow G$ be the universal covering of G , and denote $\mathfrak{k} = \mathbf{L}(K)$, $\mathfrak{p} := \text{Ker}(\theta + \text{id}_{\mathfrak{g}})$, $\tilde{K} := e^{-1}(K)$, $\tilde{\mathfrak{g}} := \mathbf{L}(\tilde{G})$, $\tilde{\mathfrak{p}} := \mathbf{L}(e)^{-1}(\mathfrak{p})$, and $\tilde{\theta} := \mathbf{L}(e)^{-1} \circ \theta \circ \mathbf{L}(e) \in \text{Aut}(\tilde{\mathfrak{g}})$. Then Proposition 4.2 shows that the mapping $\tilde{\varphi}: \tilde{K} \times \tilde{\mathfrak{p}} \rightarrow \tilde{G}$, $(\tilde{k}, \tilde{X}) \mapsto \tilde{k} \cdot \exp_{\tilde{G}} \tilde{X}$ is a diffeomorphism.

On the other hand, since the group \tilde{G} is connected and simply connected, it follows that there exists a unique smooth homomorphism $\tilde{\Theta}: \tilde{G} \rightarrow \tilde{G}$ such that $\mathbf{L}(\tilde{\Theta}) = \tilde{\theta}$. (See for instance Remark 3.13 in [Be06].) Since $\tilde{\theta}^2 = \text{id}_{\tilde{\mathfrak{g}}}$, it then follows that $\tilde{\Theta}^2 = \text{id}_{\tilde{G}}$, and in particular $\tilde{\Theta} \in \text{Aut}(\tilde{G})$.

Now we use hypothesis (iii) to see that the mapping $\tau: \mathfrak{p} \rightarrow G/K$, $X \mapsto (\exp_G X)^{-1}K$ is a diffeomorphism. In fact, it is clear that this is a smooth map, and its inverse is the smooth well-defined map $G/K \rightarrow \mathfrak{p}$, $gK \mapsto \text{pr}_{\mathfrak{p}}(g^{-1})$, where $\text{pr}_{\mathfrak{p}}: G \rightarrow \mathfrak{p}$ is the projection onto \mathfrak{p} defined by means of the diffeomorphism $\varphi^{-1}: G \rightarrow K \times \mathfrak{p}$.

Thus $\tau: \mathfrak{p} \rightarrow G/K$ is a diffeomorphism, and in particular G/K is simply connected since \mathfrak{p} is so. Then Lemma 4.1 shows that both K and \tilde{K} are connected. Since $\tilde{\theta}|_{\mathfrak{k}} = \text{id}_{\mathfrak{k}}$ and \tilde{K} is connected, it follows that $\tilde{\Theta}|_{\tilde{K}} = \text{id}_{\tilde{K}}$. In particular, the subgroup $e^{-1}(1)$ is invariant under $\tilde{\Theta}$ (since $e^{-1}(1) \subseteq \tilde{K}$). Then there exists a unique automorphism $\Theta \in \text{Aut}(G)$ such that $\Theta \circ e = e \circ \tilde{\Theta}$. In addition, since $\tilde{\Theta}^2 = \text{id}_{\tilde{G}}$, it follows that $\Theta^2 = \text{id}_G$.

It remains to prove that $K = \{g \in G \mid \Theta(g) = g\}$. The inclusion \subseteq is clear since $\tilde{\Theta}|_{\tilde{K}} = \text{id}_{\tilde{K}}$ and $K = \tilde{K}/e^{-1}(1)$. Conversely, let $g \in G$ such that $\Theta(g) = g$. By hypothesis (iii), there exist $k \in K$ and $X \in \mathfrak{p}$ such that $g = k \cdot \exp_G X$. Then $k \cdot \exp_G X = g = \Theta(g) = \Theta(k) \cdot \exp_G(\theta(-X)) = k \cdot \exp_G(-X)$, so that $\exp_G(2X) = 1$. Thus $\varphi(1, 2X) = \varphi(1, 0)$, and then $X = 0$ since $\varphi: K \times \mathfrak{p} \rightarrow G$ is injective. Consequently $g = k \in K$, and the proof ends. \square

We now come to a proposition that allows us to lift the global Iwasawa decompositions to covering groups.

Proposition 4.4. *Let G be a connected Banach-Lie group, and K , A , and N connected Banach-Lie subgroups of G such that the multiplication map $\mathbf{m}: K \times A \times N \rightarrow G$ is a diffeomorphism. In addition, assume that A and N are simply connected and $AN = NA$.*

Now assume that we have a connected Banach-Lie group \tilde{G} with a covering homomorphism $e: \tilde{G} \rightarrow G$, and define $\tilde{K} := e^{-1}(K)$, $\tilde{A} := e^{-1}(A)$, and $\tilde{N} := e^{-1}(N)$. Then \tilde{K} , \tilde{A} , and \tilde{N} are connected Banach-Lie subgroups of \tilde{G} and the multiplication map $\tilde{\mathbf{m}}: \tilde{K} \times \tilde{A} \times \tilde{N} \rightarrow \tilde{G}$ is a diffeomorphism.

Proof. It follows by Lemma 4.1(i) that \tilde{K} , \tilde{A} , and \tilde{N} are Banach-Lie subgroups of \tilde{G} and $e|_{\tilde{X}}: \tilde{X} \rightarrow X$ is a covering map when X is either of the groups K , A , N or G . Since the groups A and N are simply connected, it then follows that the map $e|_{\tilde{X}}: \tilde{X} \rightarrow X$ is actually a diffeomorphism if $X = A$ or $X = N$ (see for instance Theorem C.18 in [Be06]). In particular, the groups \tilde{A} and \tilde{N} are connected and simply connected.

To prove that \tilde{K} is connected as well, we first show that G/K is homeomorphic to the simply connected group $B := AN \simeq A \times N$, where the diffeomorphism $A \times N \rightarrow B$ is defined by the multiplication in G , as an easy consequence of the assumption. Now let $\text{pr}_K: G \rightarrow K$ and $\text{pr}_B: G \rightarrow B$ be the smooth projections given by the inverse of the diffeomorphism $K \times B \rightarrow G$. Then the continuous map $\tau: B \rightarrow G/K$, $b \mapsto b^{-1}K$ is bijective and its inverse is also continuous since it is given by $\tau^{-1}: G/K \rightarrow B$, $gK \mapsto \text{pr}_B(g^{-1})$. Thus $\tau: B \rightarrow G/K$ is a homeomorphism. By taking into account the homeomorphism $\psi: \tilde{G}/\tilde{K} \rightarrow G/K$ provided by Lemma 4.1(ii), it then follows that the quotient \tilde{G}/\tilde{K} is simply connected, since $B (\simeq A \times N)$ is simply connected. Then Lemma 4.1(iii) shows that \tilde{K} is connected.

We now prove the assertion regarding the multiplication map $\tilde{\mathbf{m}}: \tilde{K} \times \tilde{A} \times \tilde{N} \rightarrow \tilde{G}$. Using locally defined inverses of the covering map $e: \tilde{G} \rightarrow G$, we see that $\tilde{\mathbf{m}}$ is a local diffeomorphism at every point of $\tilde{K} \times \tilde{A} \times \tilde{N}$. It remains to prove that $\tilde{\mathbf{m}}$ is bijective. To prove that $\tilde{\mathbf{m}}$ is surjective, let $\tilde{g} \in \tilde{G}$

arbitrary. Since $m: K \times A \times N \rightarrow G$ is surjective, there exist $k \in K$, $a \in A$, and $n \in N$ such that $e(\tilde{g}) = kan$. Denote $\tilde{a} := (e|_{\tilde{A}})^{-1}(a) \in A$ and $\tilde{n} := (e|_{\tilde{N}})^{-1}(n) \in N$, and take $\tilde{k}_0 \in e^{-1}(k)$ arbitrary, so that $e(\tilde{k}_0 \tilde{a} \tilde{n}) = kan = e(\tilde{g})$. Denoting $\tilde{z} := \tilde{k}_0 \tilde{a} \tilde{n} \tilde{g}^{-1}$, we have $e(\tilde{z}) = 1$, so $\tilde{z} \in e^{-1}(\mathbf{1}) \subseteq e^{-1}(K) = \tilde{K}$. Then $\tilde{k} := \tilde{z}^{-1} \tilde{k}_0 \in \tilde{K}$ and $\tilde{g} = \tilde{k} \tilde{a} \tilde{n} \in \tilde{K} \tilde{A} \tilde{N}$. Since $\tilde{g} \in \tilde{G}$ is arbitrary, it follows that $\tilde{\mathbf{m}}: \tilde{K} \times \tilde{A} \times \tilde{N} \rightarrow \tilde{G}$ is surjective.

To prove that $\tilde{\mathbf{m}}$ is injective, first recall that $e|_{\tilde{X}}: \tilde{X} \rightarrow X$ is a bijective homomorphism if $X = A$ or $X = N$. Then the hypothesis that $AN = NA$ implies that $\tilde{B} := \tilde{A} \tilde{N} = \tilde{N} \tilde{A}$ is a subgroup of \tilde{G} , and the multiplication map $\tilde{A} \times \tilde{N} \rightarrow \tilde{B}$, $(\tilde{a}, \tilde{n}) \rightarrow \tilde{a} \tilde{n}$ is a bijection. Now assume that $\tilde{k}_j \in \tilde{K}$, $\tilde{a}_j \in \tilde{A}$, and $\tilde{n}_j \in \tilde{N}$ for $j = 1, 2$, and $\tilde{k}_1 \tilde{a}_1 \tilde{n}_1 = \tilde{k}_2 \tilde{a}_2 \tilde{n}_2$. Then $\tilde{k}_2^{-1} \tilde{k}_1^{-1} = \tilde{a}_2 \tilde{n}_2 (\tilde{a}_1 \tilde{n}_1)^{-1} \in \tilde{K} \cap \tilde{B}$, so that it will be enough to check that $\tilde{K} \cap \tilde{B} = \{\mathbf{1}\}$, that is, $\tilde{K} \cap \tilde{A} \tilde{N} = \{\mathbf{1}\}$. In order to prove the latter equality, let $\tilde{x} \in \tilde{K} \cap \tilde{A} \tilde{N}$ arbitrary. Then $e(\tilde{x}) \in K \cap AN = \{\mathbf{1}\}$. On the other hand, we have $\tilde{x} = \tilde{a} \tilde{n}$ for some $\tilde{a} \in \tilde{A}$ and $\tilde{n} \in \tilde{N}$, and $e(\tilde{a})e(\tilde{n}) = \mathbf{1}$. Since $A \cap N = \{\mathbf{1}\}$, it follows that $e(\tilde{a}) = e(\tilde{n}) = \mathbf{1}$. Using the fact that $e|_{\tilde{X}}: \tilde{X} \rightarrow X$ is a bijective homomorphism if $X = A$ or $X = N$, it then follows that $\tilde{a} = \tilde{n} = \mathbf{1}$, whence $\tilde{x} = \tilde{a} \tilde{n} = \mathbf{1}$. Thus $\tilde{K} \cap \tilde{A} \tilde{N} = \{\mathbf{1}\}$, and this completes the proof of the fact that the multiplication map $\tilde{\mathbf{m}}: \tilde{K} \times \tilde{A} \times \tilde{N} \rightarrow \tilde{G}$ is bijective. \square

5. WHAT A REDUCTIVE BANACH-LIE GROUP COULD BE

Reductivity relative to a symmetric norming function. This notion is suggested by Remarks 1.2 and 1.3.

Definition 5.1. Let Φ be a symmetric norming function. By Φ -reductive Lie algebra we mean any closed real Lie subalgebra of $\mathfrak{S}_{\Phi}(\mathcal{H})$ satisfying the following conditions:

- (i) for every $X \in \mathfrak{g}$ we have $X^* \in \mathfrak{g}$;
- (ii) the set $\mathfrak{g} \cap \mathfrak{F}$ of finite-rank operators in \mathfrak{g} is dense in \mathfrak{g} with respect to the norm $\|\cdot\|_{\Phi}$. Thus $\mathfrak{g} \subseteq \mathfrak{S}_{\Phi}^{(0)}$ actually.

In this case the connected Banach-Lie group $G (\subseteq \mathbf{1} + \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}))$ corresponding to the Lie subalgebra \mathfrak{g} of $\mathfrak{S}_{\Phi}^{(0)}(\mathcal{H})$ is said to be a Φ -reductive linear Banach-Lie group. By Φ -reductive Banach-Lie group we shall mean any covering group of a Φ -reductive linear Lie group.

In the above setting, the closure of $\mathfrak{g} \cap \mathfrak{F}$ with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$ will be called the L^* -algebra associated with \mathfrak{g} and will be denoted by \mathfrak{g}_2 . \square

Example 5.2. (See [dlH72] and also Definitions 3.2 and 3.2 in [Be07].) Let Φ be any symmetric norming function and denote by $\text{GL}(\mathcal{H})$ the group of all invertible bounded linear operators on the complex Hilbert space \mathcal{H} . The classical complex Banach-Lie groups and Banach-Lie algebras associated with Φ are defined as follows:

- (A) $\text{GL}_{\Phi}(\mathcal{H}) = \text{GL}(\mathcal{H}) \cap (\mathbf{1} + \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}))$ with the Lie algebra $\mathfrak{gl}_{\Phi}(\mathcal{H}) = \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H})$;
- (B) $\text{O}_{\Phi}(\mathcal{H}) = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid g^{-1} = Jg^*J^{-1}\}$ (where J is a conjugation, i.e., an antilinear isometry with $J^2 = \mathbf{1}$) with the Lie algebra $\mathfrak{o}_{\Phi}(\mathcal{H}) = \{x \in \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}) \mid x = -Jx^*J^{-1}\}$;
- (C) $\text{Sp}_{\Phi}(\mathcal{H}) = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid g^{-1} = \tilde{J}g^*\tilde{J}^{-1}\}$ (where \tilde{J} an anti-conjugation, i.e., an antilinear isometry with $\tilde{J}^2 = -\mathbf{1}$), with the Lie algebra $\mathfrak{sp}_{\Phi}(\mathcal{H}) = \{x \in \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}) \mid x = -\tilde{J}x^*\tilde{J}^{-1}\}$.

The classical real Banach-Lie groups and Banach-Lie algebras associated with the symmetric norming function Φ are the following:

- (AI) $\text{GL}_{\Phi}(\mathcal{H}; \mathbb{R}) = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid gJ = Jg\}$ with the Lie algebra $\mathfrak{gl}_{\Phi}(\mathcal{H}; \mathbb{R}) = \{x \in \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}) \mid xJ = Jx\}$, where $J: \mathcal{H} \rightarrow \mathcal{H}$ is a conjugation;
- (AII) $\text{GL}_{\Phi}(\mathcal{H}; \mathbb{H}) = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid g\tilde{J} = \tilde{J}g\}$ with the Lie algebra $\mathfrak{gl}_{\Phi}(\mathcal{H}; \mathbb{H}) = \{x \in \mathfrak{S}_{\Phi}^{(0)}(\mathcal{H}) \mid x\tilde{J} = \tilde{J}x\}$, where $\tilde{J}: \mathcal{H} \rightarrow \mathcal{H}$ is an anti-conjugation;

- (AIII) $U_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{g \in GL_\Phi(\mathcal{H}) \mid g^*Vg = V\}$ with the Lie algebra $\mathfrak{u}_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{x \in \mathfrak{S}_\Phi^{(0)}(\mathcal{H}) \mid x^*V = -Vx\}$, where $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to this orthogonal decomposition of \mathcal{H} ;
- (BI) $O_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{g \in GL_\Phi(\mathcal{H}) \mid g^{-1} = Jg^*J^{-1} \text{ and } g^*Vg = V\}$ with $\mathfrak{o}_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{x \in \mathfrak{S}_\Phi^{(0)}(\mathcal{H}) \mid x = -Jx^*J^{-1} \text{ and } x^*V = -Vx\}$, where $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to this decomposition of \mathcal{H} , and $J: \mathcal{H} \rightarrow \mathcal{H}$ is a conjugation such that $J(\mathcal{H}_\pm) \subseteq \mathcal{H}_\pm$;
- (BII) $O_\Phi^*(\mathcal{H}) = \{g \in GL_\Phi(\mathcal{H}) \mid g^{-1} = Jg^*J^{-1} \text{ and } g\tilde{J} = \tilde{J}g\}$ with the Lie algebra $\mathfrak{o}_\Phi^*(\mathcal{H}) = \{x \in \mathfrak{S}_\Phi^{(0)}(\mathcal{H}) \mid x = -Jx^*J^{-1} \text{ and } x\tilde{J} = \tilde{J}x\}$, where $J: \mathcal{H} \rightarrow \mathcal{H}$ is a conjugation and $\tilde{J}: \mathcal{H} \rightarrow \mathcal{H}$ is an anti-conjugation such that $J\tilde{J} = \tilde{J}J$;
- (CI) $Sp_\Phi(\mathcal{H}; \mathbb{R}) = \{g \in GL_\Phi(\mathcal{H}) \mid g^{-1} = \tilde{J}g^*\tilde{J}^{-1} \text{ and } gJ = Jg\}$ with the Lie algebra $\mathfrak{sp}_\Phi(\mathcal{H}; \mathbb{R}) = \{x \in \mathfrak{S}_\Phi^{(0)}(\mathcal{H}) \mid -x = \tilde{J}x^*\tilde{J}^{-1} \text{ and } xJ = Jx\}$, where $\tilde{J}: \mathcal{H} \rightarrow \mathcal{H}$ is any anti-conjugation and $J: \mathcal{H} \rightarrow \mathcal{H}$ is any conjugation such that $J\tilde{J} = \tilde{J}J$;
- (CII) $Sp_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{g \in GL_\Phi(\mathcal{H}) \mid g^{-1} = \tilde{J}g^*\tilde{J}^{-1} \text{ and } g^*Vg = V\}$ with $\mathfrak{sp}_\Phi(\mathcal{H}_+, \mathcal{H}_-) = \{x \in \mathfrak{S}_\Phi^{(0)}(\mathcal{H}) \mid x = -\tilde{J}x^*\tilde{J}^{-1} \text{ and } x^*V = -Vx\}$, where $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to this decomposition of \mathcal{H} , and $\tilde{J}: \mathcal{H} \rightarrow \mathcal{H}$ is an anti-conjugation such that $\tilde{J}(\mathcal{H}_\pm) \subseteq \mathcal{H}_\pm$.

As a by-product of the classification of the L^* -algebras (see for instance Theorems 7.18 and 7.19 in [Be06]), every (real or complex) topologically simple L^* -algebra is isomorphic to one of the classical Banach-Lie algebras associated with the Hilbert-Schmidt ideal $\mathfrak{S}_2(\mathcal{H})$.

We refer to [dlH72] and [Ne02a] for information on the homotopy groups of the classical Banach-Lie groups associated with the Schatten ideals. The corresponding description of homotopy groups actually holds true for the classical Banach-Lie groups associated with any symmetric norming function Φ . It is clear that *the connected 1-component of any classical Banach-Lie group associated with Φ is a Φ -reductive linear Banach-Lie group.* \square

It would be interesting to understand how the classical Lie algebras associated with an operator ideal fit in the framework of Φ -reductive Lie algebras. Here is a proposition in this connection. Recall that a subset $A \subseteq \mathcal{B}(\mathcal{H})$ is *irreducible* if $\{0\}$ and \mathcal{H} are the only closed linear subspaces of \mathcal{H} which are invariant under all operators in A .

Proposition 5.3. *Let Φ be a symmetric norming function, \mathfrak{g} a Φ -reductive Lie algebra, and \mathfrak{g}_2 the L^* -algebra associated with \mathfrak{g} . Then the Lie algebra \mathfrak{g} is irreducible if and only if \mathfrak{g}_2 is irreducible. In addition, if $\mathfrak{S}_\Phi \neq \mathfrak{S}_1$, then \mathfrak{g} is one of the classical (real or complex) Banach-Lie algebras associated with the norm ideal \mathfrak{S}_Φ if and only if \mathfrak{g}_2 is one of the classical (real or complex) Banach-Lie algebras associated with the Hilbert-Schmidt ideal \mathfrak{S}_2 . If this is the case, then \mathfrak{g} and \mathfrak{g}_2 are classical Lie algebras of the same type.*

Proof. Both algebras \mathfrak{g} and \mathfrak{g}_2 are closed under taking the adjoints, hence any of them is irreducible if and only if every operator that commutes with that algebra is a scalar multiple of the identity operator on \mathcal{H} . Therefore it will be enough to prove the following assertion:

$$(5.1) \quad (\forall T \in \mathcal{B}(\mathcal{H})) \quad [T, \mathfrak{g}] = \{0\} \iff [T, \mathfrak{g}_2] = \{0\}.$$

In fact, if $[T, \mathfrak{g}] = \{0\}$ then in particular $[T, \mathfrak{g} \cap \mathfrak{F}] = \{0\}$. Since the set of finite-rank operators $\mathfrak{g} \cap \mathfrak{F}$ is dense in \mathfrak{g}_2 , it follows that $[T, \mathfrak{g}_2] = \{0\}$. Conversely, if the operator T satisfies the latter condition, then we have in particular $[T, \mathfrak{g} \cap \mathfrak{F}] = \{0\}$. By property (ii) of a Φ -reductive Lie algebra, it then follows that $[T, \mathfrak{g}] = \{0\}$.

Now assume $\mathfrak{S}_\Phi \neq \mathfrak{S}_1$ and let $\tilde{\mathfrak{g}}$ be the closure of $\mathfrak{g}_2 \cap \mathfrak{F}$ with respect to the norm $\|\cdot\|_\Phi$. Since $\mathfrak{g} \cap \mathfrak{F} \subseteq \mathfrak{g}_2 \cap \mathfrak{F}$ and $\mathfrak{g} \cap \mathfrak{F}$ is dense in \mathfrak{g} , it then follows that $\mathfrak{g} \subseteq \tilde{\mathfrak{g}}$.

We are going to prove that \mathfrak{g} is actually a closed ideal of the Lie algebra $\tilde{\mathfrak{g}}$. For this purpose it will be enough to show that $[\mathfrak{g} \cap \mathfrak{F}, \mathfrak{g}_2 \cap \mathfrak{F}] \subseteq \mathfrak{g} \cap \mathfrak{F}$. In fact, let $X \in \mathfrak{g} \cap \mathfrak{F}$ and $Y \in \mathfrak{g}_2 \cap \mathfrak{F}$ arbitrary. According to the definition of \mathfrak{g}_2 , there exists a sequence $\{X_j\}_{j \geq 1}$ in $\mathfrak{g} \cap \mathfrak{F}$ such that $\|X_j - Y\|_2 \rightarrow 0$ as $j \rightarrow \infty$. Then $\|[X, X_j] - [X, Y]\|_2 \rightarrow 0$ as $j \rightarrow \infty$. On the other hand, the commutators $\{[X, X_j]\}_{j \geq 1}$ have the ranks

uniformly bounded by $\text{rank } X$, hence Lemma 4.3 in [BR05] implies that in the topology of $\|\cdot\|_\Phi$ we also have $[X, X_j] \rightarrow [X, Y]$ as $j \rightarrow \infty$, whence $[X, Y] \in \mathfrak{g} \cap \mathfrak{F}$.

Now it is easy to see that the assertion holds. In fact, if \mathfrak{g}_2 is one of the classical (real or complex) Banach-Lie algebras associated with the Hilbert-Schmidt ideal \mathfrak{S}_2 , then $\tilde{\mathfrak{g}}$ is the similar classical Banach-Lie algebra associated with the norm ideal \mathfrak{S}_Φ . Since $\mathfrak{S}_\Phi \neq \mathfrak{S}_1$, it follows as in Proposition 8 on page 92 in [dlH72] that $\tilde{\mathfrak{g}}$ has no non-trivial closed ideals, whence $\mathfrak{g} = \tilde{\mathfrak{g}}$. Conversely, if \mathfrak{g} is one of the classical (real or complex) Banach-Lie algebras associated with the norm ideal \mathfrak{S}_Φ , then it is obvious that the L^* -algebra associated with \mathfrak{g} is one of the classical L^* -algebras. \square

Group decompositions. The following theorem supplies a Cartan decomposition for some Φ -reductive Banach-Lie groups. In this connection, we refer to [Ne02b] for a systematic investigation of polar decompositions in infinite dimensions (that is, Cartan decompositions for linear groups).

Theorem 5.4. *Let Φ be a symmetric norming function and G a Φ -reductive Banach-Lie group whose Lie algebra $\mathfrak{g} \subseteq \mathfrak{S}_\Phi(\mathcal{H})$ is one of the classical Lie algebras associated with Φ . Denote $\mathfrak{k} = \{X \in \mathfrak{g} \mid X^* = -X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid X^* = X\}$, and let K be the connected Banach-Lie group corresponding to the Lie subalgebra \mathfrak{k} of \mathfrak{g} . Then K is a Banach-Lie subgroup of G and the mapping $\varphi: K \times \mathfrak{p} \rightarrow G$, $(k, X) \mapsto k \exp_G X$, is a diffeomorphism. In addition, there exists a unique automorphism $\Theta \in \text{Aut}(G)$ such that $\mathbf{L}(\Theta)X = -X^*$ for all $X \in \mathfrak{g}$ and $K = \{g \in G \mid \Theta(g) = g\}$.*

Proof. It follows by Proposition 4.2 that we may assume that G is a Φ -reductive *linear* Banach-Lie group. Thus let $G \subseteq \mathbf{1} + \mathfrak{S}_\Phi^{(0)}(\mathcal{H})$ be the connected $\mathbf{1}$ -component of the classical Banach-Lie group associated with the classical Lie algebra $\mathfrak{g} \subseteq \mathfrak{S}_\Phi(\mathcal{H})$. In this case the assertion can be proved by a method similar to the one used in Proposition III.8 of [Ne02a] in the case of the classical groups associated with the Schatten ideals. The existence of the automorphism $\Theta \in \text{Aut}(G)$ as asserted follows by an application of Proposition 4.3. \square

The following theorem states that there exist Iwasawa decompositions for the Φ -reductive Banach-Lie groups corresponding to the classical Lie algebras, provided the symmetric norming function Φ satisfies some reasonable conditions.

Theorem 5.5. *Let Φ be a mononormalizing symmetric norming function whose Boyd indices are non-trivial and let G be a Φ -reductive Banach-Lie group whose Lie algebra $\mathfrak{g} \subseteq \mathfrak{S}_\Phi(\mathcal{H})$ is one of the classical Lie algebras associated with Φ . Denote $\mathfrak{k} = \{X \in \mathfrak{g} \mid X^* = -X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid X^* = X\}$, and let K be the connected Banach-Lie group corresponding to the Lie subalgebra \mathfrak{k} of \mathfrak{g} . Then there exists $X_0 \in \mathfrak{p}$ with the following properties:*

- (a) *The set \mathcal{E}_{X_0} of spectral projections of the self-adjoint operator X_0 corresponding to the intervals of the form $(0, t]$ with $t \in \mathbb{R}$ determines a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{a}_{X_0} \dot{+} \mathfrak{n}_{X_0}$, where $\mathfrak{a}_{X_0} = \{X \in \mathfrak{p} \mid [X, X_0] = 0\}$ and $\mathfrak{n}_{X_0} = \{X \in \mathfrak{g} \cap \text{Alg}(\mathcal{E}_{X_0}) \mid X(e(\mathcal{H})) \neq e(\mathcal{H}) \text{ if } 0 \neq e \in \mathcal{E}_{X_0}\}$.*
- (b) *If we denote by A and N the connected Banach-Lie groups corresponding to the closed subalgebras \mathfrak{a}_{X_0} and \mathfrak{n}_{X_0} of \mathfrak{g} , then the multiplication map $\mathbf{m}: K \times A \times N \rightarrow G$ is a diffeomorphism. Moreover, A and N are simply connected Banach-Lie subgroups of G and $AN = NA$.*

Proof. Proposition 4.4 allows us to assume that G is actually a Φ -reductive *linear* Banach-Lie group. The idea of the proof in this case is to start by studying the group $G = \text{GL}_\Phi(\mathcal{H})$, which is the largest classical group. For this group, the construction of a local Iwasawa decomposition of its Lie algebra (assertion (a)) relies on local spectral theory (see [BS01]) along with the properties of triangular integrals established in Example 2.10. As regards the corresponding global Iwasawa decomposition (assertion (b)), one uses Corollary 2.16. See [Be07] for details. \square

Remark 5.6. The fundamental groups of the classical Banach-Lie groups associated with Φ are always abelian (see [dlH72] and [Ne02a]). It then easily follows by Remark 3.6 along with Examples 3.3 through 3.5 that all of the groups K , A , and N that occur in Theorem 5.5 are amenable, although the group G itself may not be amenable (see Example 3.5(a)). \square

Harish-Chandra decompositions. We are going to draw a little closer to representation theory, which was the main motivation of the present exposition. For this purpose we borrow the following definition of infinite-dimensional Lie groups of Harish-Chandra type from [NØ98]. Some good references for such Harish-Chandra decompositions in the setting of finite-dimensional Lie groups are [Sa80], [Kn96], and [Ne00].

Definition 5.7. By Banach-Lie group of *Harish-Chandra type* we actually mean a 4-tuple $(G, G^{\mathbb{C}}, K, K^{\mathbb{C}})$ consisting of a connected complex Banach-Lie group $G^{\mathbb{C}}$, and and three connected Banach-Lie subgroups G , K , and $K^{\mathbb{C}}$ of $G^{\mathbb{C}}$ such that the following conditions are satisfied:

- (i) the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is the complexification of the Lie algebra \mathfrak{g} of G ;
- (ii) $K^{\mathbb{C}}$ is a complex Banach-Lie subgroup of $G^{\mathbb{C}}$ and the Lie algebra $\mathfrak{k}^{\mathbb{C}}$ of $K^{\mathbb{C}}$ is the complexification of the Lie algebra \mathfrak{k} of K ;
- (iii) there exist connected complex Banach-Lie subgroups P^{\pm} of $G^{\mathbb{C}}$ whose Lie algebras \mathfrak{p}^{\pm} have the properties $(\text{ad } \mathfrak{p}^{\pm})^n \mathfrak{g}^{\mathbb{C}} = \{0\}$ for some integer $n \geq 1$ and $\mathfrak{p}^{\pm} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$, where $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ denotes the center of $\mathfrak{g}^{\mathbb{C}}$, and moreover:
 - (HC1) we have the direct sum decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{+} + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{-}$, and in addition $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{p}^{\pm}] \subseteq \mathfrak{p}^{\pm}$ and $\overline{\mathfrak{p}^{-}} = \mathfrak{p}^{+}$, where $X \mapsto \overline{X}$, $\mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$, is the antilinear involutive map whose fixed-point set is \mathfrak{g} ;
 - (HC2) the multiplication mapping $P^{+} \times K^{\mathbb{C}} \times P^{-} \rightarrow G^{\mathbb{C}}$, $(p_{+}, k, p_{-}) \mapsto p_{+}kp_{-}$, is a biholomorphic diffeomorphism onto its open image;
 - (HC3) we have $G \subseteq P^{+}K^{\mathbb{C}}P^{-}$ and $G \cap K^{\mathbb{C}}P^{-} = K$.

If the groups $G^{\mathbb{C}}$, K , and $K^{\mathbb{C}}$ are singled out in a certain context, then we may say that the group G itself is a Banach-Lie group of Harish-Chandra type. \square

Example 5.8. Let Φ be a symmetric norming function and $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$. Then the corresponding classical real Banach-Lie group of type (AIII) —in the terminology of Example 5.2— provides an example of group of Harish-Chandra type. Specifically, we mean the 4-tuple $(G, G^{\mathbb{C}}, K, K^{\mathbb{C}})$, where

$$\begin{aligned} G &= \text{U}_{\Phi}(\mathcal{H}_{+}, \mathcal{H}_{-}), \quad K = \{k \in \text{U}_{\Phi}(\mathcal{H}_{+}, \mathcal{H}_{-}) \mid k(\mathcal{H}_{\pm}) \subseteq \mathcal{H}_{\pm}\} \\ G^{\mathbb{C}} &= \text{GL}_{\Phi}(\mathcal{H}), \quad K^{\mathbb{C}} = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid g(\mathcal{H}_{\pm}) \subseteq \mathcal{H}_{\pm}\}. \end{aligned}$$

Then the conditions of Definition 5.7 are satisfied with the connected complex Banach-Lie subgroups $P^{\pm} = \{g \in \text{GL}_{\Phi}(\mathcal{H}) \mid (g - \mathbf{1})\mathcal{H}_{\mp} \subseteq \mathcal{H}_{\pm}\}$. If we write the operators on \mathcal{H} as 2×2 block matrices with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$, then

$$G \subseteq P^{+}K^{\mathbb{C}}P^{-} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_{\Phi}(\mathcal{H}) \mid D \in \text{GL}(\mathcal{H}_{-}) \right\}$$

and every element in this set can be factorized as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbf{1} & BD^{-1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ D^{-1}C & \mathbf{1} \end{pmatrix},$$

where the factors in the right-hand side belong to P^{+} , $K^{\mathbb{C}}$, and P^{-} , respectively. We refer to [NØ98] for more details as well as for similar examples provided by groups of type (BII) and (CI) (again in the terminology of Example 5.2 above). \square

Remark 5.9. As noted in [NØ98] (see also [Ne00]), the objects involved in the definition of a group of Harish-Chandra type (Definition 5.7 above) have the following additional properties:

- (a) If we denote $\mathbf{L}(P^{\pm}) = \mathfrak{p}^{\pm}$, then the exponential maps $\exp_{P^{\pm}}: \mathfrak{p}^{\pm} \rightarrow P^{\pm}$ are biholomorphic diffeomorphisms and the complex Banach-Lie groups P^{\pm} are nilpotent and simply connected. In particular there exist the logarithm maps $\log_{P^{\pm}} = (\exp_{P^{\pm}})^{-1}: P^{\pm} \rightarrow \mathfrak{p}^{\pm}$.
- (b) There exists an open connected K -invariant subset $\Omega \subseteq \mathfrak{p}^{+}$ such that the mapping $\Omega \times K^{\mathbb{C}}P^{-} \rightarrow GK^{\mathbb{C}}P^{-}$, $(Z, p) \mapsto (\exp_{G^{\mathbb{C}}} Z)p$, is a well-defined biholomorphic diffeomorphism.
- (c) Let $\zeta^{\pm}: P^{+}K^{\mathbb{C}}P^{-} \rightarrow P^{\pm}$ and $\kappa: P^{+}K^{\mathbb{C}}P^{-} \rightarrow K^{\mathbb{C}}$ be the natural projections (see condition (HC2) in Definition 5.7) and $\Xi = \{(g, Z) \in G^{\mathbb{C}} \times \mathfrak{p}^{+} \mid g \exp_{G^{\mathbb{C}}} Z \in P^{+}K^{\mathbb{C}}P^{-}\}$, which is an

open neighborhood of $G \times \Omega$ in $G^{\mathbb{C}} \times \mathfrak{p}^+$. Define $\Xi \rightarrow \mathfrak{p}^+$, $(g, Z) \mapsto g.Z = \log_{P^+}(\zeta^+(g \exp_{G^{\mathbb{C}}} Z))$ and $J: \Xi \rightarrow K$, $(g, Z) \mapsto J(g, Z) = \kappa(g \exp_{G^{\mathbb{C}}} Z)$. Then the mapping $(g, Z) \mapsto g.Z$ defines a transitive action of G upon Ω by biholomorphic diffeomorphisms and J has a cocycle property with respect to this action.

- (d) We have $0 \in \Omega$ and the isotropy group of 0 is equal to K . Thus the aforementioned action leads to a G -equivariant diffeomorphism $G/K \simeq \Omega$.

These are some of the ideas that allow one to define as in [NØ98] natural reproducing kernels on Ω out of certain representations of the Banach-Lie group K . In this way one ends up with the corresponding reproducing kernel Hilbert spaces of holomorphic functions on Ω which carry representations of the bigger Banach-Lie group G of Harish-Chandra type. \square

Remark 5.10. Applications of a different type of reproducing kernels in representation theory of Banach-Lie groups can be found in [BR07] and [BG07]. The viewpoint held in the first of these papers is somehow dual to the one of Remark 5.9 (in the sense of duality theory of symmetric spaces), i.e., the bases of the corresponding vector bundles are homogeneous spaces of compact type. The complexification of this picture of compact type is analyzed in [BG07] along with the relationship to Stinespring dilation theory, which eventually leads to geometric models for representations of operator algebras. \square

Acknowledgments. We wish to thank Professor José Galé for drawing our attention to some pertinent references, to Professor Hendrik Grundling for kindly sending us the manuscript [Gr], and to Professor Gary Weiss for comments that helped us to improve the exposition. Partial support from the grant GR202/2006 (CNCSIS code 813) is acknowledged.

REFERENCES

- [Ar78] J. Arazy, Some remarks on interpolation theorems and the boundness of the triangular projection in unitary matrix spaces, *Integral Equations Operator Theory* **1** (1978), no. 4, 453–495.
- [Ba83] W. Banaszczyk, On the existence of exotic Banach-Lie groups, *Math. Ann.* **264** (1983), no. 4, 485–493.
- [Be06] D. Beltiță, *Smooth Homogeneous Structures in Operator Theory*, Monogr. and Surveys in Pure and Appl. Math., 137. Chapman & Hall/CRC Press, Boca Raton-London-New York-Singapore, 2006.
- [Be07] D. Beltiță, Iwasawa decompositions of some infinite-dimensional Lie groups (submitted for publication). (See *preprint* math.RT/0701404.)
- [BG07] D. Beltiță, J.E. Galé, Holomorphic geometric models for representations of C^* -algebras, *preprint*, 2007.
- [BP07] D. Beltiță, B. Prunaru, Amenability, completely bounded projections, dynamical systems and smooth orbits, *Integral Equations Operator Theory* **57** (2007), no. 1, 1–17.
- [BR05] D. Beltiță, T.S. Ratiu, Symplectic leaves in real Banach Lie-Poisson spaces, *Geom. Funct. Anal.* **15** (2005), no. 4, 753–779.
- [BR07] D. Beltiță, T.S. Ratiu, Geometric representation theory for unitary groups of operator algebras, *Adv. Math.* **208** (2007), no. 1, 299–317.
- [BS01] D. Beltiță, M. Şabac, *Lie Algebras of Bounded Operators*, Operator Theory: Advances and Applications, 120. Birkhäuser Verlag, Basel, 2001.
- [Bo80] R.P. Boyer, Representation theory of the Hilbert-Lie group $U(\mathfrak{h})_2$, *Duke Math. J.* **47** (1980), no. 2, 325–344.
- [Bu50] R.C. Buck, Generalized group algebras, *Proc. Nat. Acad. Sci. USA* **36** (1950), 747–749.
- [Bu52] R.C. Buck, Operator algebras and dual spaces, *Proc. Amer. Math. Soc.* **3** (1952), 681–687.
- [DFWW04] K. Dykema, T. Figiel, G. Weiss, M. Wodzicki, Commutator structure of operator ideals, *Adv. Math.* **185**(2004), no. 1, 1–79.
- [Er72] J.A. Erdos, The triangular factorization of operators on Hilbert space, *Indiana Univ. Math. J.* **22** (1972/73), 939–950.
- [Er78] J.A. Erdos, Triangular integration on symmetrically normed ideals, *Indiana Univ. Math. J.* **27** (1978), no. 3, 401–408.
- [Ey72] P. Eymard, *Moyennes Invariantes et Représentations Unitaires*, Lecture Notes in Math. 300, Springer-Verlag, Berlin, 1972.
- [Ga06] J.E. Galé, Geometría de órbitas de representaciones de grupos y álgebras promediabiles, *Rev. R. Acad. Cienc. Exactas Fís. Quím. Nat. Zaragoza (2)* **61** (2006), 7–46.
- [GK69] I.C. Gohberg, M.G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monogr., 18, Amer. Math. Soc., Providence, RI, 1969.
- [GK70] I.C. Gohberg, M.G. Kreĭn, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. Math. Monogr., 24, Amer. Math. Soc., Providence, RI, 1970.
- [Gr] H. Grundling, unpublished manuscript.

- [Gr05] H. Grundling, Generalising group algebras, *J. London Math. Soc. (2)* **72** (2005), no. 3, 742–762.
- [GN07] H. Grundling, K.-H. Neeb, Abelian topological groups with host algebras, *J. London Math. Soc. (2)* (to appear). (See preprint math.OA/0605413.)
- [dlH72] P. de la Harpe, *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space*, Lecture Notes in Math. 285, Springer-Verlag, Berlin, 1972.
- [HR63] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis*, vol. I. Grundle. math. Wiss., Bd. 115. Springer-Verlag, Berlin, 1963.
- [KW07] V. Kaftal, G. Weiss, A survey on the interplay between arithmetic mean ideals, traces, lattices of operator ideals, and an infinite Schur-Horn majorization theorem, *preprint*, 2007.
- [Kn96] A.W. Knap, *Lie Groups Beyond an Introduction*, Progr. Math., 140, Birkhäuser-Verlag, Boston-Basel-Berlin, 1996.
- [Ne98] K.-H. Neeb, Holomorphic highest weight representations of infinite-dimensional complex classical groups, *J. reine angew. Math.* **497** (1998), 171–222.
- [Ne00] K.-H. Neeb, *Holomorphy and Convexity in Lie Theory*, Expositions in Mathematics, 28. Walter de Gruyter & Co., Berlin-New York, 2000.
- [Ne02a] K.-H. Neeb, Classical Hilbert-Lie groups, their extensions and their homotopy groups, in: *Geometry and Analysis on Finite and Infinite-Dimensional Lie Groups (Będlewo, 2000)*, Banach Center Publ., 55, Polish Acad. Sci. Warsaw, 2002, pp. 87–151.
- [Ne02b] K.-H. Neeb, A Cartan-Hadamard theorem for Banach-Finsler manifolds, *Geom. Dedicata* **95** (2002), 115–156.
- [Ne04] K.-H. Neeb, Infinite-dimensional groups and their representations, in: *Lie Theory*, Progr. Math., 228, Birkhäuser, Boston, MA, 2004, pp. 213–328.
- [Ne06] K.-H. Neeb, Towards a Lie theory of locally convex groups, *Japan. J. Math. (3rd series)* **1** (2006), no. 2, 291–468.
- [NØ98] K.-H. Neeb, B. Ørsted, Unitary highest weight representations in Hilbert spaces of holomorphic functions on infinite-dimensional domains, *J. Funct. Anal.* **156** (1998), no. 1, 263–300.
- [Pa88] A.L.T. Paterson, *Amenability*, Mathematical Surveys and Monographs, 29. American Mathematical Society, Providence, RI, 1988.
- [Pe06] V. Pestov, *Dynamics of Infinite-Dimensional Groups. The Ramsey-Dvoretzky-Milman Phenomenon*, University Lecture Series, 40. American Mathematical Society, Providence, RI, 2006.
- [Pi90] D. Pickrell, Separable representations for automorphism groups of infinite symmetric spaces, *J. Funct. Anal.* **90** (1990), no. 1, 1–26.
- [Sa80] I. Satake, *Algebraic Structures of Symmetric Domains*, Kano Mem. Lectures, 4. Publ. Math. Soc. Japan, 14. Iwanami Shoten & Princeton Univ. Press, Tokyo-Princeton, NJ, 1980.
- [SV75] Ş. Strătilă, D. Voiculescu, *Representations of AF-algebras and of the group $U(\infty)$* , Lecture Notes in Math., 486. Springer-Verlag, Berlin-New York, 1975.
- [Te63] S. Teleman, Contribution à l’analyse harmonique sur un groupe topologique quelconque, *C. R. Acad. Sci. Paris* **257** (1963), 2227–2230.
- [Up85] H. Upmeyer, *Symmetric Banach Manifolds and Jordan C^* -Algebras*, Math. Stud., 104, Notas de Mat., 96, North-Holland, Amsterdam, 1985.
- [Vo00] D.A. Vogan, Jr., The method of coadjoint orbits for real reductive groups, in: *Representation Theory of Lie Groups (Park City, 1998)*, IAS/ Park City Math. Ser. 8, Amer. Math. Soc., Providence, RI, 2000, pp. 177–238.
- [We40] A. Weil, *L’Intégration dans les Groupes Topologiques et ses Applications*, Actual. Sci. Ind., 869. Hermann et Cie., Paris, 1940.

INSTITUTE OF MATHEMATICS “SIMION STOILOW” OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, RO-014700 BUCHAREST, ROMANIA

E-mail address: Daniel.Beltita@imar.ro